Variations of Spectral Graph Isomorphism

Jeremy McMahan

December, 2018

Abstract

We continue the investigation of the spectral graph isomorphism problem (SGI). We intend to give evidence that the problem is similar in complexity to GI. To this end, we focus on showing relationships between SGI and its variations that mimic the structure of the relationships between GI and its variations.

1 Introduction

The spectral graph isomorphism problem (SGI) is an approximation version of the graph isomorphism problem (GI) where the metric of closeness between the two graphs utilizes the Rayleigh quotients of the Laplacian matrices of the graphs. In particular, we say that two simple weighted graphs, G and H, with the same vertex set are α-spectrally isomorphic if $\exists \pi \in S_n \forall x \in R^n, \frac{1}{\alpha} \leq \frac{x^T L_G x}{x^T L_{\pi(H)} x} \leq \alpha$ (Here, we assume for simplicity that $\frac{0}{0} = 1$). We note that if we replace α with 1 in the definition, we get exactly the graph isomorphism problem, so it truly is a generalization of the problem. We wish to classify the complexity of SGI. It is already known that SGI is in NP, and that the one-sided version of the problem, Graph Dominance (GD), is NP-complete. In this paper, we wish to give evidence that SGI may be close to the complexity of GI by demonstrating many properties of GI also hold for SGI. In particular, we show that reducibilities that exist between GI and variations of GI also hold for SGI and the equivalent variations of SGI. Specifically, we consider the prefix problem (PrefixSGI), coloring generalization (CSGI), and the non-trivial automorphism problem (SGA) each of which is defined in the natural way from their non-spectral counterparts. The main results are as follows:

Theorem 3.1.1. *$SGA \leq_T^p SGI$*

Theorem 3.3.1. *$PrefixSGI \cong_m^p SGI$*

Theorem 4.1.1. *$CSGI \cong_m^p SGI$*

2 Preliminaries

Let $G = (V, E, w)$ be a weighted simple graph with non-negative weights (Note we can consider loopless multi-graphs by letting the weight be the number of multi-edges between two nodes). We will refer to an edge by (u, v) even though it is undirected. Also, we will commonly think of vectors as functions $x : V \to R$ and denote the set of all such vectors R^V. Without loss of generality, we will usually assume that $V = [n]$. Then,

$$x^T L_G x = \sum_{(u,v)\in E} w_G(u, v)(x(u) - x(v))^2$$

where

$$L_G(u,v) = \begin{cases} -w_G(u,v) & \text{if } u \neq v \\ d_G(u) & \text{if } u = v \end{cases}$$

is the Laplacian matrix of G. We define the weighted degree of a vertex, u, by

$$d_G(u) = \sum_{(u,v)\in E_G} w_G(u,v)$$

We define the weighted maximum degree of G as $\Delta(G) = \max_{u\in V_G} d_G(u)$. If $S \subseteq V_G$, then we define the vector

$$\delta_S(u) = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{if } u \notin S \end{cases}$$

Lemma 2.1. $\delta_u^T L_G \delta_u = d_G(u)$

Proof:

$$\begin{aligned} \delta_u^T L_G \delta_u &= \sum_{(v,w)\in E_G} w_G(v,w)(\delta_u(v) - \delta_u(w))^2 \\ &= \sum_{(v,w)\in E_G} w_G(v,w)(0-0)^2 + \sum_{(u,v)\in E_G} w_G(u,v)(1-0)^2 \\ &= \sum_{(u,v)\in E_G} w_G(u,v) \\ &= d_G(u) \end{aligned}$$

□

Lemma 2.2. $\delta_{u,v}^T L_G \delta_{u,v} = d_G(u) + d_G(v) - 2w_G(u,v)$

Proof:

$$\begin{aligned} \delta_{u,v}^T L_G \delta_{u,v} &= \sum_{(x,y)\in E_G} w_G(x,y)(\delta_{u,v}(x) - \delta_{u,v}(y))^2 \\ &= \sum_{(x,y)\in E_G} w_G(x,y)(0-0)^2 + \sum_{(u,w)\in E_G, w\neq v} w_G(u,w)(1-0)^2 + \sum_{(v,w)\in E_G, w\neq u} w_G(v,w)(1-0)^2 \\ &\quad + w_G(u,v)(1-1)^2 \\ &= \sum_{(u,w)\in E_G, w\neq v} w_G(u,w) + \sum_{(v,w)\in E_G, w\neq u} w_G(v,w) \\ &= d_G(u) + d_G(v) - 2w_G(u,v) \end{aligned}$$

Note, if $(u,v) \notin E_G$, then $w_G(u,v) = 0$ and the two sums in the second to last line are $d_G(u), d_G(v)$ respectively, so the claim still holds in this case. □

We will use these two lemmas constantly, so don't directly refer to them by number. We also note that if G is a connected graph, then any constant vector $x(v) = c$ for all $v \in V_G$, satisfies $x^T L_G x = 0$. We denote by cH the graph H where each weight is multiplied by c. Note if $G = c\pi(H)$, then $G \cong_s^c H$. Also, we note that if π shows $G \cong_s^\alpha H$, then $\pi(H)$ must have the same components as G. Formally, if a component of G and $\pi(H)$ have a vertex in common, then these components must have the exact same vertex sets. Consequently, isolated vertices of H must be mapped to isolated vertices of G.

3 PrefixSGI

This reduction will lead to the key elements for the remaining reductions, so we start with it. Suppose we are given weighted simple graphs G and H, $\alpha \geq 1$, and a partial injective function $\psi : V_H \to V_G$. We want to know if $G \cong_s^\alpha H$ via a permutation π that preserves ψ. Intuitively, our approach is to add a gadget to the graphs so that if $\pi(i) \neq j$, yet $\psi(i) = j$, then we can find a vector x so that $\frac{x^T L_G x}{x^T L_{\pi(H)} x} > \alpha$. However, we must also ensure that if π preserves ψ and $G \cong_s^\alpha H$ is shown by this π that we can define mappings for the gadget nodes so that the graphs we construct are also α-spectrally isomorphic.

3.1 Single Mapping

First, let us consider just one mapping $\psi(i) = j$. If exactly one of $d_G(j), d_H(i)$ is 0, then no spectral isomorphism can exist preserving ψ as isolated vertices of H must be mapped to isolated vertices of G as mentioned previously. Thus, we can just output two non-isomorphic graphs and $\alpha = 1$ in this case. Note since isolated vertices have no edges incident with them and so don't affect the Rayleigh quotient, we can always arbitrarily define mappings from the isolated vertices of H to those of G. In particular, if both $d_G(j) = 0, d_H(i) = 0$, then we can always force $\pi(i) = j$, so we can assume for simplicity that no isolated vertices are mapped by ψ. Now, assume $d_G(j), d_H(i) > 0$ and so $\Delta(H) > 0$. We will add a new vertex, v_j, to both G and H and add an edge (j, v_j) to G and (i, v_j) in H. We call these new graphs G^{out} and H^{in} respectively to represent the fact that we want i, which is the input to ψ, to be mapped to j, which is the output of ψ. We will now determine the weights that will be associated with both of these edges.

First, we discuss the restrictions that are induced from the implies direction. Namely, we suppose that π preserves ψ and shows that $G \cong_s^\alpha H$. We want to then show that $G^{out} \cong_s^\alpha H^{in}$. If we define $\pi(v_j) = v_j$, we get (Note we will abuse notation and use x both for the $n+1$-dimensional vector that has a an entry for v_j and the n-dimensional vector that does not)

$$\begin{aligned}
\frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} &= \frac{x^T L_G x + w_{G^{out}}(j, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(\pi(i)) - x(\pi(v_j)))^2} \\
&= \frac{x^T L_G x + w_{G^{out}}(j, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2} \\
&\leq \frac{\alpha x^T L_{\pi(H)} x + w_{G^{out}}(j, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2} \\
&\leq \alpha \frac{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2} = \alpha
\end{aligned}$$

Where the second inequality holds if $w_{G^{out}}(j, v_j) \leq \alpha w_{H^{in}}(i, v_j)$. Also, we have

$$\begin{aligned}
\frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} &= \frac{x^T L_G x + w_{G^{out}}(j, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2} \\
&\geq \frac{\frac{1}{\alpha} x^T L_{\pi(H)} x + w_{G^{out}}(j, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2} \\
&\geq \frac{1}{\alpha} \frac{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_{H^{in}}(i, v_j)(x(j) - x(v_j))^2} = \frac{1}{\alpha}
\end{aligned}$$

So, for the second inequality to hold we need $w_{G^{out}}(j, v_j) \geq \frac{1}{\alpha} w_{H^{in}}(i, v_j)$. Note if considered other permutations than the identity on the gadgets, the restrictions just defined would be equivalent to the two length one paths, (j, v_j) and (i, v_j), being α-spectrally isomorphic. Now, we will enforce $w_{G^{out}}(j, v_j) =$

$\alpha w_{H^{in}}(i, v_j)$, which satisfies both inequalities above, and so we get that π shows $G^{out} \cong_s^\alpha H^{in}$ completing the implies direction follows. Note we chose this restriction since we will want weights to be larger in G^{out} so that we can more easily find an x satisfying $\frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} > \alpha$ when $\pi(i) \neq j$.

Now, we will look at what we need for the other direction to hold. Namely, if π witnesses that $G^{out} \cong_s^\alpha H^{in}$, then we want to make sure our new weights guarantee that $\pi(i) = j$ and the restriction of π on V_H shows $G \cong_s^\alpha H$. In particular, we will choose the weights so that $\pi(i) \neq j \implies \exists x, \frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} > \alpha$ which contradicts our assumption. Also, we will choose the weights so that $\pi(v_j) = v_j$. Let's see why this is sufficient.

First, by construction of G^{out} and H^{in},

$$\frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} = \frac{x^T L_G x + w_G(j, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_H(i, v_j)(x(\pi(i)) - x(\pi(v_j)))^2}$$

If $\pi(i) = j$ and $\pi(v_j) = v_j$, we can rewrite this equality as:

$$\frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} = \frac{x^T L_G x + w_G(j, v_j)(x(j) - x(v_j))^2}{x^T L_{\pi(H)} x + w_H(i, v_j)(j) - x(v_j)^2}$$

Now, consider any vector $x \in R^n$. We see if we extend x into an $(n+1)$-dimensional vector by setting $x(v_j) = x(j)$ that $\frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} = \frac{x^T L_G x}{x^T L_{\pi(H)} x}$. Also, we know that $\forall x \in R^{n+1}, \frac{1}{\alpha} \leq \frac{x^T L_{G^{out}} x}{x^T L_{\pi(H^{in})} x} \leq \alpha$. Thus, since $x \in R^n$ was arbitrary, we have $\forall x \in R^n, \frac{1}{\alpha} \leq \frac{x^T L_G x}{x^T L_{\pi(H)} x} \leq \alpha$.

There are two cases to consider to ensure $\pi(i) = j$,

- Suppose $\pi(i) \neq j$ and $\pi(v_j) \neq j$. Let us consider the ratio of Rayleigh quotients with respect to the vector δ_j. We note that $d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j)$ since (j, v_j) is the only new edge incident to j in G^{out}. Consequently, $\delta_j^T L_{G^{out}} \delta_j = d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j) > w_{G^{out}}(j, v_j)$ since $d_G(j) > 0$. Now, since $\pi(\pi^{-1}(j)) = j$, j in $\pi(H^{in})$ corresponds to $\pi^{-1}(j)$ in H^{in}. In particular, we have $d_{\pi(H^{in})}(j) = d_{H^{in}}(\pi^{-1}(j))$. Also, since the only edge in H^{in} that is not in H is incident to i and v_j, and $\pi^{-1}(j) \notin \{i, v_j\}$ by assumption, we know $d_{\pi(H^{in})}(j) = d_{H^{in}}(\pi^{-1}(j)) = d_H(\pi^{-1}(j)) \leq \Delta(H)$. Thus, $\delta_j^T L_{\pi(H^{in})} \delta_j = d_{\pi(H^{in})}(j) \leq \Delta(H)$. Consequently,

$$\frac{\delta_j^T L_{G^{out}} \delta_j}{\delta_j^T L_{\pi(H^{in})} \delta_j} = \frac{d_{G^{out}}(j)}{d_{\pi(H^{in})}(j)} = \frac{d_G(j) + w_{G^{out}}(j, v_j)}{d_H(\pi^{-1}(j))} \geq \frac{d_G(j) + w_{G^{out}}(j, v_j)}{\Delta(H)} > \frac{w_{G^{out}}(j, v_j)}{\Delta(H)}$$

  Hence, to enforce the ratio of Rayleigh quotients is greater than α, it suffices that $w_{G^{out}}(j, v_j) \geq \alpha \Delta(H)$.

- Next, let's consider the case where $\pi(v_j) = j$. In this case, we again consider the vector δ_j. As before, $\delta_j^T L_{G^{out}} \delta_j = d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j) > w_{G^{out}}(j, v_j)$. Also, $\pi(v_j) = j \implies \pi^{-1}(j) = v_j$, so $\delta_j^T L_{\pi(H^{in})} \delta_j = d_{\pi(H^{in})}(j) = d_{H^{in}}(v_j) = w_{H^{in}}(i, v_j)$ being that (i, v_j) is the only edge incident to v_j in H^{in} by definition. Hence,

$$\frac{\delta_j^T L_{G^{out}} \delta_j}{\delta_j^T L_{\pi(H^{in})} \delta_j} = \frac{d_{G^{out}}(j)}{d_{\pi(H^{in})}(j)} = \frac{d_G(j) + w_{G^{out}}(j, v_j)}{w_{H^{in}}(i, v_j)} > \frac{w_{G^{out}}(j, v_j)}{w_{H^{in}}(i, v_j)} = \frac{\alpha w_{H^{in}}(i, v_j)}{w_{H^{in}}(i, v_j)} = \alpha$$

  where we used the fact that we forced $w_{G^{out}}(j, v_j) = \alpha w_{H^{in}}(i, v_j)$. Hence, no further restrictions on the weights arise from this case.

Lastly, we will want $\pi(i) = j$ implies $\pi(v_j) = v_j$ as well. Since π is a bijection, we know $\pi(v_j) \neq j$. So, suppose $\pi(v_j) = k$. We will consider the vector δ_{v_j}. Since (j, v_j) is the only edge incident to v_j in

G^{out} by definition, we know $\delta_{v_j}^T L_{G^{out}} \delta_{v_j} = d_{G^{out}}(v_j) = w_{G^{out}}(j, v_j)$. Also, $\pi(v_j) = k \implies \pi^{-1}(v_j) = k$, so $\delta_{v_j}^T L_{\pi(H^{in})} \delta_{v_j} = d_{\pi(H^{in})}(v_j) = d_H(k)$. Consequently,

$$\frac{\delta_{v_j}^T L_{G^{out}} \delta_{v_j}}{\delta_{v_j}^T L_{\pi(H^{in})} \delta_{v_j}} = \frac{d_{G^{out}}(v_j)}{d_{\pi(H^{in})}(v_j)} = \frac{w_{G^{out}}(j, v_j)}{d_H(k)} \geq \frac{w_{G^{out}}(j, v_j)}{\Delta(H)}$$

Hence, to guarantee the ratio of Rayleigh quotients is strictly greater than α, it suffices that $w_{G^{out}}(j, v_j) > \alpha\Delta(H)$.

Thus, choosing $w_{G^{out}}(j, v_j) = 2\alpha\Delta(H)$, we get strict inequalities in each case. Consequently, we see that $\pi(i) = j$ and $\pi(v_j) = v_j$ otherwise our assumption that π demonstrates that $G^{out} \cong_s^\alpha H^{in}$ is contradicted. Then, as before, we have π shows $G \cong_s^\alpha H$ and $\pi(i) = j$. This completes the reduction for the single mapping case and its proof.

Theorem 3.1.1. *SGA* $\leq_T^p$ *SGI*

3.2 Technical Lemmas

The remaining reductions will use similar structure to the one above, so we shall state the corresponding technical lemmas here. Specifically, for the implies direction of the reductions, we will use the following lemma. Intuitively, this lemma says that if we look at the union of two graphs with no edges in common, then the α-spectrally isomorphic relation will be preserved. For example, in the previous argument for the implies direction of the one mapping case of PrefixSGI, we had that G' was the edge-less graph in addition to the edge (j, v_j) and H' was the edge-less graph with edge (i, v_j).

Lemma 3.2.1. If G, G' and H, H' are edge-disjoint graphs, π shows that $G \cong_s^\alpha H$, π' shows that $G' \cong_s^\alpha H'$, and $\forall v \in V_H \cap V_{H'}, \pi(v) = \pi'(v)$, then π'' shows

$$G \cup G' \cong_s^\alpha H \cup H'$$

where $\pi'' = \pi$ on V_H and $\pi'' = \pi'$ on $V_{H'}$

Proof: Since G, G' are edge-disjoint, we know $E_{G \cup G'} = E_G \dot{\cup} E_{G'}$, so

$$\begin{aligned}
x^T L_{G \cup G'} x &= \sum_{(u,v) \in E_{G \cup G'}} w_{G \cup G'}(u, v)(x(u) - x(v))^2 \\
&= \sum_{(u,v) \in E_G \dot{\cup} E_{G'}} w_{G \cup G'}(x(u) - x(v))^2 \\
&= \sum_{(u,v) \in E_G} w_G(u, v)(x(u) - x(v))^2 + \sum_{(u,v) \in E_{G'}} w_{G'}(u, v)(x(u) - x(v))^2 \\
&= x^T L_G x + x^T L_{G'} x
\end{aligned}$$

Also, since π and π' match on vertices common to both H and H', we have that π'' is a permutation of $V_{H \cup H'}$. Consequently, since H, H' are edge-disjoint, $E_{\pi''(H \cup H')} = \pi''(E_{H \cup H'}) = \pi''(E_H \cup E_{H'}) = \pi''(E_H) \cup \pi''(E_{H'})$. Thus, we also have $x^T L_{\pi''(H \cup H')} x = x^T L_{\pi''(H)} x + x^T L_{\pi''(H')} x$. Putting these together, we have:

$$\begin{aligned}
x^T L_{G \cup G'} x &= x^T L_G x + x^T L_{G'} x \\
&\leq \alpha x^T L_{\pi(H)} x + \alpha x^T L_{\pi'(H')} x && \text{Assumption on } \pi, \pi' \\
&= \alpha(x^T L_{\pi''(H)} x + x^T L_{\pi''(H')} x) && \text{Definition of } \pi'' \\
&= \alpha x^T L_{\pi''(H \cup H')} x
\end{aligned}$$

Similarly,

$$\begin{aligned}
x^T L_{G\cup G'} x &= x^T L_G x + x^T L_{G'} x \\
&\geq \frac{1}{\alpha} x^T L_{\pi(H)} x + \frac{1}{\alpha} x^T L_{\pi'(H')} x && \text{Assumption on } \pi, \pi' \\
&= \frac{1}{\alpha} (x^T L_{\pi''(H)} x + x^T L_{\pi''(H')} x) && \text{Definition of } \pi'' \\
&= \frac{1}{\alpha} x^T L_{\pi''(H\cup H')} x
\end{aligned}$$

Hence, π'' shows that $G \cup G' \cong_s^\alpha H \cup H$. □

The next lemma will give us the machinery necessary to tackle the implied by direction. In particular, the lemma shows that under certain conditions, if we attach gadgets to the vertices of two graphs, then it will not affect if the original graphs were α-spectrally isomorphic.

Lemma 3.2.2. Suppose G, G' and H, H' are edge-disjoint graphs and π shows $G \cup G' \cong_s^\alpha H \cup H'$. Also, suppose each component of G' has exactly one vertex of G and let G'_v denote the component incident with $v \in V_G$, and similarly for H'. Furthermore, suppose $\pi(V_H \cap V_{H'}) = V_G \cap V_{G'}$ and $\pi(V_{H'_v}) = V_{G'_{\pi(v)}}$ for all $v \in V_H \cap V_{H'}$, then π (specifically the restriction of π on V_H) shows

$$G \cong_s^\alpha H$$

Proof: First, since G, G' and H, H' are edge disjoint, we know

$$\frac{x^T L_{G\cup G'} x}{x^T L_{\pi(H\cup H')} x} = \frac{x^T L_G + xT L_{G'} x}{x^T L_{\pi(H)} x + x^T L_{\pi(H')} x}$$

Also, since each component of G' and H' are incident to exactly one vertex of G and H respectively, we can partition the Laplacian of G' as

$$x^T L_{G'} x = \sum_{v\in V_G \cap V_{G'}} x^T L_{G'_v} x$$

and similarly for H'. Furthermore, applying π to H' just applies π to each component of H', so

$$x^T L_{\pi(H')} x = \sum_{v\in V_H \cap V_{H'}} x^T L_{\pi(H'_v)} x$$

Now, since $\pi(V_H \cap V_{H'}) = V_G \cap V_{G'}$, we have $v \in V_G \cap V_{G'} \iff \pi^{-1}(v) \in V_H \cap V_{H'}$. Hence, we can rewrite the previous equality as

$$x^T L_{\pi(H')} x = \sum_{v\in V_G \cap V_{G'}} x^T L_{\pi(H'_{\pi^{-1}(v)})} x$$

Putting all these equalities together then gives

$$\frac{x^T L_{G\cup G'} x}{x^T L_{\pi(H\cup H')} x} = \frac{x^T L_G + \sum_{v\in V_G\cap V_{G'}} x^T L_{G'_v} x}{x^T L_{\pi(H)} x + \sum_{v\in V_G\cap V_{G'}} x^T L_{\pi(H'_{\pi^{-1}(v)})} x}$$

Now, if we consider any vector x with $x(w) = x(v)$ for each $w \in V_{G'_v}$, then x is just a constant vector over $V_{G'_v}$, so we know that $x^T L_{G'_v} x = 0$. Also, by assumption, we have $\pi(V_{H'_{\pi^{-1}(v)}}) = V_{G'_v}$, so for this x we also know $x^T L_{\pi(H'_{\pi^{-1}(v)})} x = 0$. Consequently, if we consider all vectors x so that $x(v)$ is arbitrary for any $v \in V_G$, and $x(w) = x(v)$ for all $w \in V_{G'_v}$ where $v \in V_G \cap V_{G'}$, we get

$$\frac{x^T L_{G\cup G'} x}{x^T L_{\pi(H\cup H')} x} = \frac{x^T L_G x + \sum_{v\in V_G\cap V_{G'}} x^T L_{G'_v} x}{x^T L_{\pi(H)} x + \sum_{v\in V_G\cap V_{G'}} x^T L_{\pi(H'_{\pi^{-1}(v)})} x} = \frac{x^T L_G x}{x^T L_{\pi(H)} x}$$

Since π shows $G \cup G' \cong_s^\alpha H \cup H'$, we know for all such x,

$$\frac{1}{\alpha} \leq \frac{x^T L_{G\cup G'} x}{x^T L_{\pi(H \cup H')} x} = \frac{x^T L_G x}{x^T L_{\pi(H)} x} \leq \alpha$$

Lastly, since each $x \in R^{V_G}$ can be uniquely extend into a vector of the form above, we have

$$\frac{1}{\alpha} \leq \frac{x^T L_G x}{x^T L_{\pi(H)} x} \leq \alpha$$

holds for all $x \in R^{V_G}$.

□

3.3 Multiple Mappings

Now, we consider the case when more than one mapping is defined. The critical difference is now we could have just mapped all of the inputs of ψ to all of the outputs of ψ, but not in the correct way. Specifically, suppose $\pi(i) = k$, where k is something else that is mapped to in ψ, i.e. $\psi(\ell) = k$, and $\pi(\ell) = r$ where r is something else mapped to and so on. The most difficult case is if two numerically adjacent vertices are swapped, so we just consider this case. Specifically, say $\pi(i) = j$ and $\pi(\ell) = j-1$, but $\psi(i) = j-1, \psi(\ell) = j$. We consider the vector δ_j. Again, $\delta_j^T L_{G^{out}} \delta_j = d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j) > w_{G^{out}}(j, v_j)$. Now, since $\pi(i) = j$, j in $\pi(H^{in})$ corresponds to i in H^{in}. In particular, we have $d_{\pi(H^{in})}(j) = d_{H^{in}}(i)$. Also, since $\psi(i) = j-1$, we have the edge (i, v_{j-1}) is incident to i in H^{in} in addition to the edges incident to i in H by construction. Consequently, $d_{H^{in}}(i) = d_H(i) + w_{H^{in}}(i, v_{j-1}) \leq \Delta(H) + w_{H^{in}}(i, v_{j-1})$, so $\delta_j^T L_{\pi(H^{in})} \delta_j \leq \Delta(H) + w_{H^{in}}(i, v_{j-1})$. Lastly, we will still ensure that $w_{G^{out}}(j-1, v_{j-1}) = \alpha w_{H^{in}}(i, v_{j-1})$. Putting this all together we see,

$$\frac{\delta_j^T L_{G^{out}} \delta_j}{\delta_j^T L_{\pi(H^{in})} \delta_j} = \frac{d_{G^{out}}(j)}{d_{\pi(H^{in})}(j)} = \frac{d_G(j) + w_{G^{out}}(j, v_j)}{d_H(i) + w_{H^{in}}(i, v_{j-1})} > \frac{w_{G^{out}}(j, v_j)}{\Delta(H) + w_{H^{in}}(i, v_{j-1})} = \frac{w_{G^{out}}(j, v_j)}{\Delta(H) + \frac{1}{\alpha} w_{G^{out}}(j-1, v_{j-1})}$$

We want the last quantity to be at least α, and so we want $w_{G^{out}}(j, v_j) \geq \alpha\Delta(H) + w_{G^{out}}(j-1, v_{j-1})$. Now, noting that we already derived the restriction that each new weight in G^{out} is $\geq 2\alpha\Delta(H)$ to ensure $\pi(v_j) \neq k$ for any k, we get a recursive inequality for the weights: $w_{G^{out}}(1, v_1) \geq 2\alpha\Delta(H)$ and for $j \geq 2, w_{G^{out}}(j, v_j) \geq \alpha\Delta(H) + w_{G^{out}}(j-1, v_{j-1})$. Solving this recurrence leads to: $w_{G^{out}}(j, v_j) \geq (j+1)\alpha\Delta(H)$. Now we summarize the final reduction from PrefixSGI to SGI:

Theorem 3.3.1. $PrefixSGI \cong_m^p SGI$

$SGI \leq_m^p PrefixSGI$ is immediate, so we show $PrefixSGI \leq_m^p SGI$.

Proof: Given G, H, α, ψ, for each mapping $\psi(i) = j$, we add a vertex v_j to both G and H and edges (j, v_j) and (i, v_j) with weights $(j+1)\alpha\Delta(H)$ and $(j+1)\Delta(H)$, respectively, and call these new graphs G^{out} and H^{in}. Our reduction outputs are then G^{out}, H^{in}, and α. As before, if any mapping $\psi(i) = j$ satisfies $d_H(i) > 0 \oplus d_G(j) > 0$, then this constraint can't be satisfied, so we just output two non-isomorphic graphs and $\alpha = 1$. Also, recall if $d_H(i) = 0, d_G(j) = 0$, then since values of these vertices don't affect the Rayleigh quotients, we can always force $\pi(i) = j$, so we will for simplicity assume $d_H(i) > 0, d_G(j) > 0$ throughout the proof. Consequently, we can also assume $\Delta(H) > 0$.

[$\Longrightarrow$] Suppose π preserves ψ and shows $G \cong_s^\alpha H$. We note that $G^{out} = G \cup G'$ where G' is the disjoint union of weighted edges (j, v_j) and isolated vertices $v \in V_G \setminus Range(\psi)$. Formally, $G' = ([n] \cup \{v_j | j \in Range(\psi)\}, \{(j, v_j) | j \in Range(\psi)\})$ with $w_{G'}(j, v_j) = (j+1)\alpha\Delta(H)$. Also, $H^{in} = H \cup H'$ where H'

is the disjoint union of weighted edges $(i, v_{\psi(i)})$ and isolated vertices $v \in V_H \setminus Domain(\psi)$. Formally, $H' = ([n] \cup \{v_j | j \in Range(\psi)\}, \{(i, v_{\psi(i)}) | i \in Domain(\psi)\})$ with $w_{H'}(i, v_{\psi(i)}) = (\psi(i)+1)\Delta(H)$. Define π' to be π on $V_H \cap V_{H'} = [n]$ and the identity otherwise. In particular, we have $\pi'(v_j) = v_j$ for any gadget vertex $v_j \in V_{H'}$ and $\pi'(i) = \pi(i)$ for any non-gadget vertex $i \in V_{H'}$. Then, $\alpha\pi'(H') = G'$ since for any edge $(i, v_{\psi(i)}) \in E_{H'}$, $\pi'((i, v_{\psi(i)})) = (\pi(i), id(v_{\psi(i)})) = (\psi(i), v_{\psi(i)}) \in E_{G'}$ being that π preserves ψ and by construction

$$\alpha w_{\pi(H')}(\psi(i), v_{\psi(i)}) = \alpha w_{H'}(i, v_{\psi(i)}) = \alpha w_{H^{in}}(i, v_{\psi(i)}) = w_{G^{out}}(\psi(i), v_{\psi(i)}) = w_{G'}(\psi(i), v_{\psi(i)})$$

Thus, π' shows $G' \cong_s^\alpha H'$. Also, by construction, π and π' match on $V_H \cap V_{H'} = [n]$. Thus, Lemma 3.2.1 gives $G^{out} = G \cup G' \cong_s^\alpha H \cup H' = H^{in}$.

[$\Longleftarrow$] Suppose π shows $G^{out} \cong_s^\alpha H^{in}$. Note that $d_G(j) = 0 \oplus d_{\pi(H)}(j) = 0$ can never happen since this would imply the components of G and $\pi(H)$ differ contradicting that π shows $G^{out} \cong_s^\alpha H^{in}$. Hence, all vertex degrees we consider below are all positive since we are assuming for simplicity that isolated vertices aren't mapped in ψ.

Lemma 3.3.2. π preserves ψ

Proof: We proceed by contradiction and utilize the well ordering principle. Suppose π does not preserve ψ and consider the largest j for which $\pi(i) \neq j$ but $\psi(i) = j$. We will consider the three possibilities for where j ends up in $\pi(H^{in})$, i.e. $\pi^{-1}(j)$.

- If $\pi(\ell) = j$ and $\ell \in [n] \setminus Domain(\psi)$, then consider vector, δ_j. Recall, $d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j) = d_G(j) + (j+1)\alpha\Delta(H)$ by construction of G^{out}. Now, since $\pi(\ell) = j$, $d_{\pi(H^{in})}(j) = d_{H^{in}}(\ell)$. Also, since $\ell \notin Domain(\psi)$, we know ℓ is not incident to any of the new edges not in H by construction, so $d_{H^{in}}(\ell) = d_H(\ell)$. Consequently,

$$\frac{\delta_j^T L_{G^{out}} \delta_j}{\delta_j^T L_{\pi(H^{in})} \delta_j} = \frac{d_{G^{out}}(j)}{d_{\pi(H^{in})}(j)} = \frac{d_G(j) + (j+1)\alpha\Delta(H)}{d_H(\ell)} > \frac{(j+1)\alpha\Delta(H)}{\Delta(H)} = (j+1)\alpha > \alpha$$

- If $\pi(\ell) = j$ and $\psi(\ell) = k$ (so $\ell \in Domain(\psi)$), then by maximality of j we know that $k < j$, i.e. $k+1 \leq j$. Consider the vector δ_j. Again, $d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j) = d_G(j) + (j+1)\alpha\Delta(H)$ by construction of G^{out}. Now, since $\pi(\ell) = j$, $d_{\pi(H^{in})}(j) = d_{H^{in}}(\ell)$. Also, since $\ell \in Domain(\psi)$, we know ℓ is incident to a new edge not in H by construction, so $d_{H^{in}}(\ell) = d_H(\ell) + w_{H^{in}}(\ell, v_k) = d_H(\ell) + (k+1)\Delta(H)$. Hence,

$$\frac{\delta_j^T L_{G^{out}} \delta_j}{\delta_j^T L_{\pi(H^{in})} \delta_j} = \frac{d_{G^{out}}(j)}{d_{\pi(H^{in})}(j)} = \frac{d_G(j) + (j+1)\alpha\Delta(H)}{d_H(\ell) + (k+1)\Delta(H)} > \frac{(k+2)\alpha\Delta(H)}{\Delta(H) + (k+1)\Delta(H)} = \frac{(k+2)\alpha\Delta(H)}{(k+2)\Delta(H)} = \alpha$$

- If $\pi(v_k) = j$, there are two further cases to consider:
  - If $k \leq j$, then again consider the vector δ_j. Once more, $d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j) = d_G(j) + (j+1)\alpha\Delta(H)$ by construction of G^{out}. Now, since $\pi(v_k) = j$, $d_{\pi(H^{in})}(j) = d_{H^{in}}(v_k) = (k+1)\Delta(H)$ by construction. Thus,

  $$\frac{\delta_j^T L_{G^{out}} \delta_j}{\delta_j^T L_{\pi(H^{in})} \delta_j} = \frac{d_{G^{out}}(j)}{d_{\pi(H^{in})}(j)} = \frac{d_G(j) + (j+1)\alpha\Delta(H)}{(k+1)\Delta(H)} > \frac{(j+1)\alpha\Delta(H)}{(k+1)\Delta(H)} \geq \alpha$$

  - If $k > j$, then by maximality of j, we know that if $\psi(\ell) = k$ then $\pi(\ell) = k$. Hence, we consider the vector $\delta_{k,j}$. Since $j, k \in Range(\psi)$, $d_{G^{out}}(j) = d_G(j) + w_{G^{out}}(j, v_j) = d_G(j) + (j+1)\alpha\Delta(H)$ and $d_{G^{out}}(k) = d_G(k) + w_{G^{out}}(k, v_k) = d_G(k) + (k+1)\alpha\Delta(H)$. Also, $w_{G^{out}}(j,k) = w_G(j,k)$ since any edge in G is the same in G^{out} by construction. Thus, $\delta_{k,j}^T L_{G^{out}} \delta_{k,j} = d_{G^{out}}(j) + d_{G^{out}}(k) - 2w_{G^{out}}(j,k) = d_G(j) + d_G(k) - 2w_G(j,k) + (j+1)\alpha\Delta(H) + (k+1)\alpha\Delta(H)$. Now, we know that

$\pi(\ell) = k$ and $\pi(v_k) = j$, so the edge (ℓ, v_k) in H^{in} becomes (k, j) in $\pi(H^{in})$. Consequently, $d_{\pi(H^{in})}(k) = d_H(\ell) + w_{H^{in}}(\ell, v_k)$ and $d_{\pi(H^{in})}(j) = d_{H^{in}}(v_k) = w_{H^{in}}(\ell, v_k)$, so $\delta_{k,j}^T L_{\pi(H^{in})}\delta_{k,j} = d_{\pi(H^{in})}(j) + d_{\pi(H^{in})}(k) - 2w_{\pi(H^{in})}(j,k) = (d_H(\ell) + w_{H^{in}}(\ell, v_k)) + w_{H^{in}}(\ell, v_k) - 2w_{H^{in}}(\ell, v_k) = d_H(\ell)$. So,

$$\frac{\delta_{k,j}^T L_{G^{out}}\delta_{k,j}}{\delta_{k,j}^T L_{\pi(H^{in})}\delta_{k,j}} = \frac{d_G(j) + d_G(k) - 2w_G(j,k) + (j+1)\alpha\Delta(H) + (k+1)\alpha\Delta(H)}{d_H(\ell)} > \frac{(k+1)\alpha\Delta(H)}{\Delta(H)} > \alpha$$

Thus, in any case we reach a contradiction of the assumption that π shows $G^{out} \cong_s^\alpha H^{in}$, so the claim holds. □

Lemma 3.3.3. If π preserves ψ, then π is the identity on the gadget vertices

Proof: Suppose not and let j be the largest vertex satisfying $\pi(v_j) \neq v_j$. Since π preserves ψ by Lemma 3.3.2 we know that nothing in the domain of ψ can map to v_j, so there are only two cases to consider:

- If $\pi(k) = v_j$ where k is not in the domain of ψ, then consider δ_{v_j}. By construction of G^{out}, $d_{G^{out}}(v_j) = w_{G^{out}}(j, v_j) = (j+1)\alpha\Delta(H)$. Now, since $\pi(k) = v_j$, $d_{\pi(H^{in})}(v_j) = d_{H^{in}}(k)$. Also, since $k \notin Domain(\psi)$, we know k is not incident to any of the new edges not in H by construction, so $d_{H^{in}}(k) = d_H(k)$. Consequently,

$$\frac{\delta_{v_j}^T L_{G^{out}}\delta_{v_j}}{\delta_{v_j}^T L_{\pi(H^{in})}\delta_{v_j}} = \frac{d_{G^{out}}(v_j)}{d_{\pi(H^{in})}(v_j)} = \frac{(j+1)\alpha\Delta(H)}{d_H(k)} \geq \frac{(j+1)\alpha\Delta(H)}{\Delta(H)} = (j+1)\alpha > \alpha$$

- If $\pi(v_k) = v_j$, then by assumption of maximality, we have $k < j$ (since otherwise we'd have $\pi(v_k) \neq v_k$ yet $k > j$), so $j \geq k+1$. Now, consider the vector δ_{k,v_j}. We know $k \in Range(\psi)$ since v_k exists, so suppose $\psi(\ell) = k$. First, we know that (k, v_j) is not an edge of G^{out}, so $w_{G^{out}}(k, v_j) = 0$. Consequently, $\delta_{k,v_j}^T L_{G^{out}}\delta_{k,v_j} = d_{G^{out}}(k) + d_{G^{out}}(v_j) = (d_G(k) + (k+1)\alpha\delta(H)) + (j+1)\alpha\Delta(H)$. Now, we know that $\pi(v_k) = v_j$ and $\pi(\ell) = k$, so the edge (ℓ, v_k) in H^{in} becomes (k, v_j) in $\pi(H^{in})$. Consequently, $d_{\pi(H^{in})}(k) = d_H(\ell) + w_{H^{in}}(\ell, v_k)$ and $d_{\pi(H^{in})}(v_j) = d_{H^{in}}(v_k) = w_{H^{in}}(\ell, v_k)$, so $\delta_{k,v_j}^T L_{\pi(H^{in})}\delta_{k,v_j} = d_{\pi(H^{in})}(v_j) + d_{\pi(H^{in})}(k) - 2w_{\pi(H^{in})}(v_j, k) = (d_H(\ell) + w_{H^{in}}(\ell, v_k)) + w_{H^{in}}(\ell, v_k) - 2w_{H^{in}}(\ell, v_k) = d_H(\ell)$.

$$\frac{\delta_{k,v_j}^T L_{G^{out}}\delta_{k,v_j}}{\delta_{k,v_j}^T L_{\pi(H^{in})}\delta_{k,v_j}} = \frac{d_G(k) + (j+1)\alpha\Delta(H) + (k+1)\alpha\Delta(H)}{d_H(\ell)} > \frac{(j+k+2)\alpha\Delta(H)}{\Delta(H)} > \alpha$$

Thus, in either case we reach a contradiction of the assumption that π shows $G^{out} \cong_s^\alpha H^{in}$, so the claim holds. □

Hence, we have that ψ is preserved by π by Lemma 3.3.2 and π is the identity on the gadgets by Lemma 3.3.3. As mentioned previously, we have $G^{out} = G \cup G'$ and $H^{in} = H \cup H'$ except now we will not have the isolated vertices in G' and H'. Formally, $G' = (Range(\psi) \cup \{v_j | j \in Range(\psi)\}, \{(j, v_j) | j \in Range(\psi)\})$ with $w_{G'}(j, v_j) = (j+1)\alpha\Delta(H)$ and Formally, $H' = (Domain(\psi) \cup \{v_j | j \in Range(\psi)\}, \{(i, v_{\psi(i)}) | i \in Domain(\psi)\})$ with $w_{H'}(i, v_{\psi(i)}) = (\psi(i)+1)\Delta(H)$. Now, we note that by construction $V_H \cap V_{H'} = Domain(\psi)$ and $V_G \cap V_{G'} = Range(\psi)$. Consequently, Since π preserves ψ, we have $\pi(V_H \cap V_{H'}) = \pi(Domain(\psi)) = Range(\psi) = V_G \cap V_{G'}$. Also, each component of H' is just a single edge $(i, v_{\psi(i)})$ containing a single vertex $i \in V_H$. Each such edge forms the component V_i' in the setting of Lemma 3.2.2. Similarly, each component of G' is just a single edge (j, v_j) containing a single vertex $j \in V_G$. Since π is the identity on the gadget vertices and preserves ψ, $\pi(V_{H_i'}) = \pi(\{i, v_{\psi(i)}\}) = \{\psi(i), v_{\psi(i)}\} = V_{G'_{\psi(i)}} = V_{G'_{\pi(i)}}$. Thus, by Lemma 3.2.2 we have π shows $G \cong_s^\alpha H$ and by Lemma 3.3.2, π preserves ψ.

4 Coloring

4.1 CSGI

We approach CSGI similarly to PrefixSGI. Now, we have a coloring function $c : V_H \to [n]$ that we need to preserve. We will attach a gadget to every vertex in both graphs. Specifically, for every j, we add edges (j, v_j) to both G and H to form G^c and H^c. The idea is we want to assign the same weights to the added edges for each vertex in the same color class so that they can be interchanged with each other. However, we will ensure weights of other classes are different in order to get a ratio of Rayleigh quotients that is too large in the case where we map a vertex of one color to another. Before, we considered preimages, namely, where did j end up. Here, it is more natural to look at images, namely, where does i go. We will end up using well-ordering, so there are two relevant cases to consider.

- Suppose $\pi(i) = j$ and $c(i) < c(j)$. Consider the vector δ_j. Since each vertex j of G is incident to the edge (j, v_j) in G^c, $d_{G^c}(j) = d_G(j) + w_{G^c}(j, v_j)$ by construction of G^c. Now, since $\pi(i) = j$, $d_{\pi(H^c)}(j) = d_{H^c}(i) = d_H(i) + w_{H^c}(i, v_i)$ by construction of H^c. Consequently,

$$\frac{\delta_j^T L_{G^c} \delta_j}{\delta_j^T L_{\pi(H^c)} \delta_j} = \frac{d_{G^c}(j)}{d_{\pi(H^c)}(j)} = \frac{d_G(j) + w_G(j, v_j)}{d_H(i) + w_H(i, v_i)} \geq \frac{w_{G^c}(j, v_j)}{\Delta(H) + w_{H^c}(i, v_i)}$$

  We want the last term to be $> \alpha$. This leads to the condition that $w_{G^c}(j, v_j) > \alpha\Delta(H) + w_{G^c}(i, v_i)$

- Suppose $\pi(i) = v_j$ and $c(i) < c(j)$. Now, we consider δ_{v_j}.

$$\frac{\delta_{v_j}^T L_{G^c} \delta_{v_j}}{\delta_{v_j}^T L_{\pi(H^c)} \delta_{v_j}} = \frac{w_{G^c}(j, v_j)}{d_H(i) + w_{H^c}(i, v_i)} \geq \frac{w_{G^c}(j, v_j)}{\Delta(H) + w_{H^c}(i, v_i)}$$

  Again, we want the last term to be $> \alpha$. This leads to the inequality $w_{G^c}(j, v_j) > \alpha\Delta(H) + w_{G^c}(i, v_i)$. Hence, it is sufficient that $w_{G^c}(j, v_j) = 2\alpha\Delta(H) + w_{G^c}(i, v_i)$. Now, we want that the weights of each vertex in the same color class is the same, so we can rewrite this equation more generally as $w_{G^c}(\text{class } c(j)) = 2\alpha\Delta(H) + w_{G^c}(\text{class } c(i))$. We no longer have any condition on individual weights, just how they relate to one another, so we can choose $w_{G^c}(\text{class } 1)$ to be an arbitrary non-negative number. To give us a fairly clean reduction, we choose $w_{G^c}(\text{class } 1) = 2\alpha\Delta(H)$. Solving this recurrence gives $w_{G^c}(\text{class } i) = 2i\alpha\Delta(H)$. In other words, $w_{G^c}(i, v_i) = 2c(i)\alpha\Delta(H)$

Now we show the reduction is correct.

Theorem 4.1.1. *CSGI* $\cong_m^p$ *SGI*

SGI $\leq_m^p$ *CSGI* is immediate, so we show *CSGI* $\leq_m^p$ *SGI*.

Proof:

Given G, H, α, c, we construct graphs G^c, H^c, where G^c is G in addition to edges (i, v_i) with weights $2c(i)\alpha\Delta(H)$ and H^c is H in addition to edges (i, v_i) with weights $2c(i)\Delta(H)$. The output of the reduction is then G^c, H^c, α. If $\Delta(H) = 0$, then the only way $G \cong_s^\alpha H$ regardless of c is if G is also the edgeless graph. Hence, we will not add gadgets in that case. Thus, we assume that $\Delta(H) > 0$.

[$\Longrightarrow$] Suppose π preserves color and shows $G \cong_s^\alpha H$. We note that $G^c = G \cup G'$ where $G' = ([n] \cup \{v_i | i \in [n]\}, \{(i, v_i) | i \in [n]\})$ with $w_{G'}(i, v_i) = 2c(i)\alpha\Delta(H)$. Also, $H^c = H \cup H'$ where $H' = ([n] \cup \{v_i | i \in [n]\}, \{(i, v_i) | i \in [n]\})$ with $w_{H'}(i, v_i) = 2c(i)\Delta(H)$. Define π' to be π on $V_H \cap V_{H'} = [n]$ and $\pi'(v_i) = v_{\pi(i)}$ for each $i \in [n]$. Then, $\alpha\pi'(H') = G'$ since for any edge $(i, v_i) \in E_{H'}$, $\pi'((i, v_i)) = (\pi(i), \pi(v_i)) = (\pi(i), v_{\pi(i)}) \in E_{G'}$ by construction. Also, since π preserves color, $c(i) = c(\pi(i))$, so

$$\alpha w_{\pi(H')}(\pi(i), v_{\pi(i)}) = \alpha w_{H'}(i, v_i) = \alpha 2c(i)\Delta(H) = 2c(\pi(i))\alpha\Delta(H) = w_{G'}(\pi(i), v_{\pi(i)})$$

Thus, π' shows $G' \cong_s^\alpha H'$ being that $\alpha\pi(H') = G'$. Now, by construction, π and π' match on $V_H \cap V_{H'} = [n]$. Thus, Lemma 3.2.1 gives $G^c = G \cup G' \cong_s^\alpha H \cup H' = H^c$.

[$\Longleftarrow$] Suppose π shows that $G^c \cong_s^\alpha H^c$.

Lemma 4.1.2. π preserves colors

Proof: Suppose not and let i be the smallest vertex for which $c(\pi(i)) \neq c(i)$. There are three main cases to consider

- if $\pi(i) = j$ and $c(i) \neq c(j)$, then by assumption that i is the smallest vertex not having its color preserved, we know $c(i) < c(j)$. Now, we consider δ_j. Since each vertex j of G is incident to the edge (j, v_j) in G^c, $d_{G^c}(j) = d_G(j) + w_{G^c}(j, v_j) = d_G(j) + 2c(j)\alpha\Delta(H)$ by construction of G^c. Now, since $\pi(i) = j$, $d_{\pi(H^c)}(j) = d_{H^c}(i) = d_H(i) + w_{H^c}(i, v_i) = d_H(i) + 2c(i)\Delta(H)$. Consequently,

$$\frac{\delta_j^T L_{G^c} \delta_j}{\delta_j^T L_{\pi(H^c)} \delta_j} = \frac{d_{G^c}(j)}{d_{\pi(H^c)}(j)} = \frac{d_G(j) + 2c(j)\alpha\Delta(H)}{d_H(i) + 2c(i)\Delta(H)} \geq \frac{2c(j)\alpha\Delta(H)}{(2c(i)+1)\Delta(H)} \geq \frac{2(c(i)+1)\alpha}{2c(i)+1} > \alpha$$

- $\pi(i) = v_j$ and $c(i) < c(j)$, then consider δ_{v_j}. By construction, v_j is only incident to the edge (j, v_j) in G^c. Hence, $d_{G^c}(v_j) = w_{G^c}(j, v_j) = 2c(j)\alpha\Delta(H)$. Now, since $\pi(i) = v_j$, $d_{\pi(H^c)}(v_j) = d_{H^c}(i) = d_H(i) + w_{H^c}(i, v_i) = d_H(i) + 2c(i)\Delta(H)$. Hence,

$$\frac{\delta_{v_j}^T L_{G^c} \delta_{v_j}}{\delta_{v_j}^T L_{\pi(H^c)} \delta_{v_j}} = \frac{d_{G^c}(v_j)}{d_{\pi(H^c)}(v_j)} = \frac{2c(j)\alpha\Delta(H)}{d_H(i) + 2c(i)\Delta(H)} \geq \frac{2c(j)\alpha\Delta(H)}{(2c(i)+1)\Delta(H)} \geq \frac{2(c(i)+1)\alpha}{2c(i)+1} > \alpha$$

- $\pi(i) = v_j$ and $c(i) \geq c(j)$, then there are two further cases to consider.
  - If $\pi(v_i) = j$ and $c(i) = c(j)$, so i and v_i have been shifted and swapped, then consider δ_j if $d_G(j) > 0$.

$$\frac{\delta_j^T L_{G^c} \delta_j}{\delta_j^T L_{\pi(H^c)} \delta_j} = \frac{d_{G^c}(j)}{d_{\pi(H^c)}(j)} = \frac{d_G(j) + 2c(j)\alpha\Delta(H)}{2c(i)\Delta(H)} > \frac{2c(i)\alpha\Delta(H)}{2c(i)\Delta(H)} = \alpha$$

    However, if $d_H(i) = 0 = d_G(j)$, then only the edge incident to j in $\pi(H^c)$ is (j, v_j). Hence, j only contributes $w_{H^c}(i, v_i)(x(j) - x(v_j))^2$ to $x^T L_{\pi(H^c)} x$. Consequently, we can just consider the new permutation π' that is π except $\pi'(i) = j, \pi'(v_i) = v_j$. Since $w_{H^c}(i, v_i)(x(\pi'(i)) - x(\pi'(v_i)))^2 = w_{H^c}(i, v_i)(x(j) - x(v_j))^2 = w_{H^c}(i, v_i)(x(v_j) - x(j))^2 = w_{H^c}(i, v_i)(x(\pi(i)) - x(\pi(v_i))^2$ and π' agrees with π on all other vertices, $x^T L_{\pi(H^c)} x = x^T L_{\pi'(H^c)} x$. Hence, π' also shows $G^c \cong_s^\alpha H^c$, yet has i's color preserved since $\pi'(i) = j$ and by assumption $c(i) = c(j)$. We can then continue the proof using π' instead of π, so we can without loss of generality assume this case does not occur.
  - If $\pi(v_i) \neq j$ or $c(i) > c(j)$, then we note that in the latter case when $c(i) > c(j)$, we know by minimality of i, that $c(\pi(j)) = c(j)$. Hence, $\pi(v_i) \neq j$ being that they have different colors. Consequently, we just assume $\pi(v_i) \neq j$. We consider $\delta_{v_j, \pi(v_i)}$. Since $\pi(v_i) \neq j$, and only j is adjacent to v_j in G^c, we know that $\pi(v_i)$ and v_j are not adjacent in G^c, so $w_{G^c}(v_j, \pi(v_i)) = 0$. So, $\delta_{v_j,\pi(v_i)}^T L_{G^c} \delta_{v_j,\pi(v_i)} = d_{G^c}(v_j) + d_{G^c}(\pi(v_i))$. Next, we know every vertex is incident to a weighted edge, and each weighted edge is at least $2\alpha\Delta(H)$ in G^c, so $d_{G^c}(\pi(v_i)) \geq 2\alpha\Delta(H)$. Thus, $\delta_{v_j,\pi(v_i)}^T L_{G^c} \delta_{v_j,\pi(v_i)} = d_{G^c}(v_j) + d_{G^c}(\pi(v_i)) \geq 2c(j)\alpha\Delta(H) + 2\alpha\Delta(H)$. Now, we know that $\pi(i) = v_j$, so the edge (i, v_i) in H^c becomes $(v_j, \pi(v_i))$ in $\pi(H^c)$. Therefore, $d_{\pi(H^c)}(v_j) = d_H(i) + w_{H^c}(i, v_i)$ and $d_{\pi(H^{in})}(\pi(v_i)) = d_{H^c}(v_i) = w_{H^c}(i, v_i)$, so $\delta_{v_j,\pi(v_i)}^T L_{\pi(H^c)} \delta_{v_j,\pi(v_i)} = d_{\pi(H^c)}(v_j) + d_{\pi(H^c)}(\pi(v_i))) - 2w_{\pi(H^{in})}(v_j, k) = (d_H(i) + w_{H^c}(i, v_i)) + w_{H^c}(i, v_i) - 2w_{H^c}(i, v_i) = d_H(i)$. Hence,

$$\frac{\delta_{v_j,\pi(v_i)}^T L_{G^c} \delta_{v_j,\pi(v_i)}}{\delta_{v_j,\pi(v_i)}^T L_{\pi(H^c)} \delta_{v_j,\pi(v_i)}} = \frac{d_{G^c}(v_j) + d_{G^c}(\pi(v_i))}{d_{\pi(H^c)}(v_j) + d_{\pi(H^c)}(\pi(v_i)) - 2w_{H^c}(i, v_i)} \geq \frac{2c(j)\alpha\Delta(H) + 2\alpha\Delta(H)}{\Delta(H)} > \alpha$$

Hence, in all cases, we found a vector that makes the ratio of Rayleigh quotients $> \alpha$ contradicting our assumption that π shows $G^c \cong_s^\alpha H^c$. Thus, π must preserve color. □

Lemma 4.1.3. $\forall i, j \in [n], \pi(i) = j \implies \pi(v_i) = v_j$, i.e. $\pi(v_i) = v_{\pi(i)}$

Proof: Suppose there exists i, j with $\pi(i) = j$, but $\pi(v_i) \neq v_j$ consider the vector $\delta_{j,\pi(v_i)}$. The argument is almost identical to before. Since $\pi(v_i) \neq v_j$, we have by definition that $\delta_{j,\pi(v_i)}(v_j) = 0$ and $\delta_{j,\pi(v_i)}(j) = 1$. Thus, the term $w_{G^c}(j, v_j)(x(j) - x(v_j))^2$ from $\delta^T_{j,\pi(v_i)} L_{G^c} \delta_{j,\pi(v_i)}$ becomes $w_{G^c}(j, v_j)$. Hence, $\delta^T_{j,\pi(v_i)} L_{G^c} \delta_{j,\pi(v_i)} \geq w_{G^c}(j, v_j) = 2c(j)\alpha\Delta(H)$. Now, we know that $\pi(i) = j$, so the edge (i, v_i) in H^c becomes $(j, \pi(v_i))$ in $\pi(H^c)$. Therefore, $d_{\pi(H^c)}(j) = d_H(i) + w_{H^c}(i, v_i)$ and $d_{\pi(H^{in})}(\pi(v_i)) = d_{H^c}(v_i) = w_{H^c}(i, v_i)$, so $\delta^T_{v_j,\pi(v_i)} L_{\pi(H^c)} \delta_{v_j,\pi(v_i)} = d_{\pi(H^c)}(v_j) + d_{\pi(H^c)}(\pi(v_i))) - 2w_{\pi(H^{in})}(v_j, k) = (d_H(i) + w_{H^c}(i, v_i)) + w_{H^c}(i, v_i) - 2w_{H^c}(i, v_i) = d_H(i)$. Thus,

$$\frac{\delta^T_{j,\pi(v_i)} L_{G^c} \delta_{j,\pi(v_i)}}{\delta^T_{j,\pi(v_i)} L_{\pi(H^c)} \delta_{j,\pi(v_i)}} \geq \frac{w_{G^c}(j, v_j)}{d_H(i)} \geq \frac{2c(j)\alpha\Delta(H)}{\Delta(H)} > \alpha$$

Hence, we found a vector that makes the ratio of Rayleigh quotients $> \alpha$ contradicting our assumption that π shows $G^c \cong_s^{al} H^c$. Thus, $\pi(v_i) = v_{\pi(i)}$ for all i. □

As mentioned previously, we have $G^c = G \cup G'$ and $H^c = H \cup H'$. Now, we note that by construction $V_H \cap V_{H'} = V_G \cap V_{G'} = [n]$. Consequently, Since π preserves ψ, we have $\pi(V_H \cap V_{H'}) = \pi([n]) = [n] = V_G \cap V_{G'}$. Also, each component of H' is just a single edge (i, v_i) containing a single vertex $i \in V_H$. Each such edge forms the component V_i' in the setting of Lemma 3.2.2. Similarly, each component of G' is just a single edge (i, v_i) containing a single vertex $i \in V_G$. By Lemma 4.1.3, $\pi(V_{H_i'}) = \pi(\{i, v_i\}) = \{\pi(i), v_{\pi(i)}\} = V_{G'_{\pi(i)}}$. Thus, by Lemma 3.2.2 we have π shows $G \cong_s^\alpha H$ and by Lemma 4.1.2, π preserves color.