Spectral Graph Isomorphism

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January 14, 2018

Abstract

We continue the investigation of the spectral graph isomorphism problem (SGI). We mainly focus on new developments that result from looking at the problem from the perspective of a decision problem. To begin, we show general properties of the spectrally isomorphic relation. Also, we show general combinatorial conditions that guarantee or prevent graphs from being spectrally isomorphic with respect to particular functions. In addition, we show certain graphs to be spectrally isomorphic with specific bounds. Furthermore, we show lower bounds on the functions for which two graphs may be spectrally isomorphic. Lastly, we inspect the computational complexity of SGI.

1 Definitions

Let $G = (V, E_G)$ and $H = (V, E_H)$ be graphs with the same vertex set. We say that G and H are α-spectrally isomorphic, if there exists a permutation $\pi : V \to V$ so that G and $\pi(H)$ have the same components, equivalently, the null space of the Laplacians are the same, and $\forall x \in R^V \setminus N(L_G)$, $\frac{1}{\alpha} \leq \frac{x^T L_G x}{x^T L_{\pi(H)} x} \leq \alpha$, where $\pi(H)$ is the graph $(\pi(V), E_H)$. Equivalently, we define $\pi(H) = (V, \{(\pi^{-1}(u), \pi^{-1}(v)) | (u, v) \in E_H\})$, which merely takes an edge, (u,v), of H and puts the same edge in the permuted graph where the endpoints, u and v, of the edge are now the vertices that we identify with u and v under the permutation. This characterization will be more useful in this paper as it immediately shows that H and $\pi(H)$ are isomorphic with isomorphism π^{-1}.

Generally, we view α as some function of $n = |V|$. We say that G and H are spectrally isomorphic if there is some α so that G and H are α-spectrally isomorphic. Sometimes, we think of the spectral graph isomorphism problem as the optimization problem that seeks the smallest α so that G and H are α-spectrally isomorphic. Equivalently, it seeks $\min_{\pi \in Sym(V)} \kappa(G, \pi(H))$, where $\kappa(G, H) = \max\{\max_x \frac{x^T L_G x}{x^T L_H x}, \max_x \frac{x^T L_H x}{x^T L_G x}\}$ is the relative condition number of G and H. However, more often in this paper, we will treat the spectral graph isomorphism problem as a variation of a decision problem. Specifically, given some α, are G and H α-spectrally isomorphic?

2 Spectral Isomorphism

2.1 The Spectral Isomorphism Relation

To begin this section, we introduce the notation $G \cong_s^\alpha H$ to mean that G and H are α-spectrally isomorphic. Also, we write $G \cong_s H$ if G and H are spectrally isomorphic, equivalently, there is some α so that $G \cong_s^\alpha H$. More generally, if F is a set of functions then we say that G and H are F-spectrally isomorphic if $G \cong_s^f H$

for some $f \in F$ and denote this by $G \cong_F H$. We will commonly use the subscripts C, L, P to denote the set of constant, logarithmic, and polynomial functions respectively. Furthermore, we say that $G \cong_s^\alpha H$ is optimal if α is a solution to SGI for G and H. Note that we always have that $G \cong_s^{\kappa(G,H)} H$ and the relative condition number between two graphs with the same components is always at most a polynomial in n and so $G \cong_s H \iff G \cong_P H$.

Also, we say that a graph G and a graph H with the same vertices have the same component structure, if there exists a bijection g from the components of G to the components of H such that for any component, K, of G we have $|V_K| = |V_{g(K)}|$. In other words, G and H have the same number of components having the same number of vertices.

Lemma 2.1. *$G \cong_s H \iff$ G and H have the same component structure*

Proof.

- [$\implies$] We have $G \cong_s H \iff$ G and $\pi(H)$ have the the same components for some permutation $\pi \iff$ G and $\pi(H)$ have the same component structure $\implies$ G and H have the same component structure since permuting the vertices of H cannot change the order of each component, just the names of each component's vertices.
- [$\impliedby$] Suppose g proves that G and H have the same component structure. Let K be a component of G and consider $g(K)$. We define a permutation π that arbitrarily maps the vertices of $g(K)$ to the vertices of K. This is always possibly since the number of vertices of K and $g(K)$ are the same by definition. We notice that G and $\pi(H)$ now have the same components, so as previously mentioned $G \cong_s \pi(H)$. Hence, $G \cong_s H$ by simply composing the two permutations.

□

Now, we prove a few simple lemmas in order to establish that $\cong_s$ is an equivalence relation.

Lemma 2.2. *For any permutation π, if Π is the permutation matrix representing π defined by $\Pi(u, v) = [\pi^{-1}(u) = v]$, then $(\Pi x)(a) = x(\pi^{-1}(a))$ and two graphs G, H are isomorphic with isomorphism π if and only if $\Pi A_G \Pi^T = A_H$.*

Proof. By [4] □

Lemma 2.3. *For any graph G and any permutation, π, $\Pi^T L_G \Pi = L_{\pi(G)}$ where Π is the permutation matrix representing π as in Lemma 2.2.*

Proof. As mentioned in the definitions section, we have that G is isomorphic to $\pi(G)$ with isomorphism π^{-1}. Now, if Π is defined as in Lemma 2.2 with respect to π, then Π^T is the permutation matrix representing π^{-1}. Specifically, we have $\Pi^T(u, v) = \Pi(v, u) = [\pi^{-1}(v) = u] = [\pi(u) = v] = [(\pi^{-1})^{-1}(u) = v]$. Hence, by Lemma 2.2, we know that $(\Pi^T)A_G(\Pi^T)^T = \Pi^T A_G \Pi = A_{\pi(G)}$. Also, we have that $\Pi^T D_G \Pi = D_{\pi(G)}$. In particular, since π^{-1} is an isomorphism from G to $\pi(G)$, we have that $\forall u \in V, d_G(u) = d_{\pi(G)}(\pi^{-1}(u))$ and so $\forall u \in V, d_G(\pi(u)) = d_{\pi(G)}(u)$. Consequently, $\forall u \in V, D_{\pi(G)}(u, u) = d_{\pi(G)}(u) = d_G(\pi(u))$. In addition,

$\forall u \in V$,

$$\begin{aligned}
(\Pi^T D_G \Pi)(u,u) &= \sum_{w\in V}\sum_{y\in V} \Pi^T(u,y) D_G(y,w)\Pi(w,u) && \text{Matrix Multiplication}\\
&= \sum_{w\in V}\sum_{y\in V} [\pi(u)=y] D_G(y,w)[\pi^{-1}(w)=u] && \text{Definition of } \Pi, \Pi^T\\
&= \sum_{w\in V} [\pi(u)=w] D_G(w,w)[\pi^{-1}(w)=u] && D \text{ is diagonal}\\
&= D_G(\pi(u),\pi(u)) = d_G(\pi(u)) = D_{\pi(G)}(u,u)
\end{aligned}$$

Hence, since both matrices are diagonal and have the same diagonals, we know that $\Pi^T D_G \Pi = D_{\pi(G)}$. Thus, by definition of the laplacian, we have $\Pi^T L_G \Pi = \Pi^T(D_G - A_G)\Pi = \Pi^T D_G \Pi - \Pi^T A_G \Pi = D_{\pi(G)} - A_{\pi(G)} = L_{\pi(G)}$. □

Theorem 2.4. *$\cong_s$ is an equivalence relation. Moreover, $\cong_F$ is an equivalence relation for any set of functions, F, that forms a group under multiplication.*

Proof.

1. Reflexive. The identity permutation satisfies that G and $id(G) = G$ are 1-spectrally isomorphic, so reflexivity holds. Also, since 1 is the multiplicative identity, which is contained in any set of functions that form a group under multiplication, we see that $\cong_F$ is reflexive.

2. Symmetric. Suppose $G \cong_s^\alpha H$, then there is some permutation π so that G and $\pi(H)$'s Laplacions have the same null space and $\forall x \in R^V \setminus N(L_G)$, $\frac{1}{\alpha} \le \frac{x^T L_G x}{x^T L_{\pi(H)} x} \le \alpha$.

   Then we have $\forall x \in R^V \setminus N(L_G)$,

$$\begin{aligned}
\frac{x^T L_G x}{x^T L_{\pi(H)} x} &= \frac{x^T L_G x}{x^T \Pi^T L_H \Pi x} && \text{By Lemma 2.3}\\
&= \frac{x^T \Pi^T \Pi L_G \Pi^T \Pi x}{x \Pi^T L_H \Pi x} && \text{Since } \Pi^{-1} = \Pi^T\\
&= \frac{(\Pi x)^T \Pi L_G \Pi^T (\Pi x)}{(\Pi x)^T L_H (\Pi x)} && \text{Since } (AB)^T = B^T A^T\\
&= \frac{(\Pi x)^T L_{\pi^{-1}(G)} (\Pi x)}{(\Pi x)^T L_H (\Pi x)} && \text{Since } \Pi^T \text{ represents } \pi^{-1}\\
&= \frac{y^T L_{\pi^{-1}(G)} y}{y^T L_H y} && \text{By letting } y = \Pi x
\end{aligned}$$

   Also, we have $x \notin N(L_G) \iff x \notin N(L_{\pi(H)}) \iff x \notin N(\Pi^T L_H \Pi) \iff \Pi x \notin N(L_H) \iff y = \Pi x \notin N(L_H)$ and so the above equality holds $\forall y \notin N(L_H)$. Now, by taking the original inequalities and flipping them, we see that $\forall y \in R^V \setminus N(L_H)$, $\frac{1}{\alpha} \leq \frac{y^T L_H y}{y^T L_{\pi^{-1}(G)} y} \leq \alpha$. In addition, we have

$$\begin{aligned}
x \in N(L_H) &\iff L_H x = 0 && \text{By definition of Null Space}\\
&\iff \Pi^T L_H x = 0 && \text{Since } \Pi^T \text{ is invertible}\\
&\iff \Pi^T L_H \Pi \Pi^T x = 0 && \text{Since } \Pi^T = \Pi^{-1}\\
&\iff L_{\pi(H)} \Pi^T x = 0 && \text{By Lemma 2.3}\\
&\iff \Pi^T x \in N(L_{\pi(H)}) && \text{By definition of Null Space}\\
&\iff \Pi^T x \in N(L_G) && \text{Since } N(L_G) = N(L_{\pi(H)})\\
&\iff L_G \Pi^T x = 0 && \text{By definition of Null Space}\\
&\iff \Pi L_G \Pi^T x = 0 && \text{Since } \Pi \text{ is invertible}\\
&\iff L_{\pi^{-1}(G)} x = 0 && \text{By Lemma 2.3}\\
&\iff x \in N(L_{\pi^{-1}(G)}) && \text{By definition of Null Space}
\end{aligned}$$

Hence, H and $L_{\pi^{-1}(G)}$ have the same null space. Thus,$H \cong_s^\alpha G$, so symmetry holds. Since the bound is the same and $\alpha \in F$ implies $\alpha \in F$, we see that $\cong_F$ is also symmetric.

3. Transitive. Suppose $G \cong_s^\alpha H$ and $H \cong_s^\beta K$, then there exist permutations π_1, π_2 so that $N(L_G) = N(L_{\pi_1(H)})$ and $N(L_H) = N(L_{\pi_2(K)})$ and $\forall x \in R^V \setminus N(L_G)$, $\frac{1}{\alpha} \leq \frac{x^T L_G x}{x^T L_{\pi_1(H)} x} \leq \alpha$ and $\forall x \in R^V \setminus N(L_H)$, $\frac{1}{\beta} \leq \frac{x^T L_H x}{x^T L_{\pi_2(K)} x} \leq \beta$.

   Then $\forall x \in R^V \setminus N(L_H)$,

$$\begin{aligned}
\frac{x^T L_H x}{x^T L_{\pi_2(K)} x} &= \frac{x^T \Pi_1 \Pi_1^T L_H \Pi_1 \Pi_1^T x}{x^T \Pi_1 \Pi_1^T L_{\pi_2(K)} \Pi_1 \Pi_1^T x} && \text{Since } \Pi_1^{-1} = \Pi_1^T\\
&= \frac{x^T \Pi_1 L_{\pi_1(H)} \Pi_1^T x}{x^T \Pi_1 L_{\pi_1 \pi_2(K)} \Pi_1^T x} && \text{By Lemma 2.3}\\
&= \frac{(\Pi_1^T x)^T L_{\pi_1(H)} (\Pi_1^T x)}{(\Pi_1^T x)^T L_{\pi_1 \pi_2(K)} (\Pi_1^T x)} && \text{Since } (AB)^T = B^T A^T\\
&= \frac{y^T L_{\pi_1(H)} y}{y^T L_{\pi_1 \pi_2(K)} y} && \text{By letting } y = \Pi_1^T x
\end{aligned}$$

   Also, we have $x \notin N(L_H) \iff \Pi_1^T x \notin N(\Pi_1^T L_H \Pi) \iff \Pi_1^T x \notin N(L_{\pi_1(H)}) \iff y = \Pi_1^T x \notin N(L_{\pi_1(H)})$ and so the above equality holds $\forall y \notin N(L_{\pi_1(H)})$. However, since $N(L_G) = N(L_{\pi_1(H)})$, we have the equality holds $\forall y \notin N(L_G)$.

   Thus, we know that $\forall y \in R^V \setminus N(L_G)$, $\frac{1}{\beta} \leq \frac{y^T L_{\pi_1(H)} y}{y^T L_{\pi_1 \pi_2(K)} y} \leq \beta$.

   Hence, $\forall z \in R^V \setminus N(L_G)$, $\frac{z^T L_G z}{z^T L_{\pi_1(H)} z} \times \frac{z^T L_{\pi_1(H)} z}{z^T L_{\pi_1 \pi_2(K)} z} = \frac{z^T L_G z}{z^T L_{\pi_1 \pi_2(K)} z}$. Now, by multiplying all the inequalities together we see that $\forall x \in R^V \setminus N(L_G)$,

$$\frac{1}{\alpha\beta} \leq \frac{x^T L_G x}{x^T L_{\pi_1 \pi_2(K)} x} \leq \alpha\beta$$

   Now, we argue that $N(L_G) = N(L_{\pi_1 \pi_2(K)})$, by showing that $N(L_{\pi_1(H)}) = N(L_{\pi_1 \pi_2(K)})$.

$$
\begin{aligned}
x \in N(L_{\pi_1(H)}) &\iff L_{\pi_1(H)}x = 0 && \text{By definition of Null Space}\\
&\iff \Pi_1^T L_H \Pi_1 x = 0 && \text{By Lemma 2.3}\\
&\iff L_H \Pi_1 x = 0 && \text{Since } \Pi_1^T \text{ is invertible}\\
&\iff \Pi_1 x \in N(L_H) && \text{By definition of Null Space}\\
&\iff \Pi_1 x \in N(L_{\pi_2(K)}) && \text{Since } N(L_H) = N(L_{\pi_2(K)})\\
&\iff L_{\pi_2(K)}\Pi_1 x = 0 && \text{By definition of Null Space}\\
&\iff \Pi_1^T L_{\pi_2(K)}\Pi_1 x = 0 && \text{Since } \Pi^T \text{ is invertible}\\
&\iff L_{\pi_1\pi_2(K)}x = 0 && \text{By Lemma 2.3}\\
&\iff x \in N(L_{\pi_1\pi_2(K)}) && \text{By definition of Null Space}
\end{aligned}
$$

Hence, by transitivity of equality we have $N(L_G) = N(L_{\pi_1\pi_2(K)})$. Thus, $G \cong_s^{\alpha\beta} K$, so transitivity holds. Now, if $\alpha, \beta \in F$, then since F is closed under multiplication, being that it is a group under multiplication, we see that $G \cong_F H$, so $\cong_F$ is transitive.

□

We will frequently use the stronger facts exhibited by the proof that symmetry of the relation actually preserves the function and transitivity merely multiplies the two functions. In fact, we can show a further strengthening.

Corollary 2.4.1. *$G \cong_s^\alpha H$ is optimal $\iff$ $H \cong_s^\alpha G$ is optimal.*

Proof. We show the implication, the converse is identical. We proceed by contradiction. Suppose that $G \cong_s^\alpha H$ is optimal, yet $H \cong_s^\beta G$ for some $\beta < \alpha$. Then, we have by symmetry of $\cong_s$ that $G \cong_s^\beta H$, which contradicts the minimality of α. □

However, cannot derive a similar strengthening for transitivity of this form being that it does not hold in general (almost never, in fact).

2.2 Complexity Theoretic Perspective

We define the spectral isomorphism class of a function $\alpha(n)$, where n is the number of vertices of graphs we are considering, as $SI(\alpha) = \{(G, H) | G \cong_s^\alpha H\}$. Also, for a set of functions, F, we define the F-spectral isomorphism class, FSI, as the union over all functions $f \in F$ of $SI(f)$. Equivalently, this is the set of all pairs of graphs that are F-spectrally isomorphic. In particular, we will use the same letters as in Section 2.1 to denote the more common classes of functions. Specifically, we consider CSI = SI(O(1)), LSI = SI(O(logn)), and PSI = SI($n^{O(1)}$). Also, notice that $PSI = \cup_\alpha SI(\alpha)$ encapsulates all graphs that are spectrally isomorphic since $G \cong_s H \iff G \cong_P H$. Naturally, we are more interested in the constant spectral isomorphism class, as these graphs have more in common.

Another important set we will consider is the α-edge difference class, $ED(\alpha)$, where $ED(\alpha) = \{(G, H)|$ G and H have the same component structure and $E_G \Delta E_H = \alpha(n)\}$. Also, for any set of functions F, we define $FED = \cup_{f\in F} ED(f)$. In addition, if $(G, H) \in FED$, then we say that G and H differ by F-many edges. Also, for any graph G, we let FED(G) be the set of all graphs with the same components as G that differ in F-many edges from G. Similarly, for each class, C, defined, we define C_s to be the restriction of C over pairs of graphs with the same components. Note that any two graphs with n vertices can differ by at most $\binom{n}{2}$ edges and so PED is exactly the set of all pairs of graphs of n vertices with the same component structure. Thus, $PED = PSI$. We explore further relationships between theses classes later on.

3 Optimization Lemmas

In this section, we establish several optimization results that will be later useful in deriving functions for which two graphs may be spectrally isomorphic.

Definition 3.1. *Let G be a graph, $S \subseteq E_G$, and $T \subseteq V_G$. Then, $V(S) = \{u|(u,v) \in S\}$ is the set of all vertices of G incident to S. Also, we say some set of subgraphs of G, K, is T-internally disjoint if each subgraph has only vertices of T in common, i.e. $\forall H, J \in K, V_H \cap V_J \subseteq T$.*

Definition 3.2. *Let G be a graph, $S \subseteq E_G$, and $H = G - S$. Then, we define $C(H)$ to be the set of all subgraphs of H formed by taking a component, K, of $G - V(S)$ that has edges to both u and v in H for some $(u,v) \in S$ and adding to it each vertex of $V(S)$ with edges to K in H along with those edges. Also, we let $CS(H) = \cup_{K \in C} K$.*

The idea behind defining C(H) is it abstracts away much of the parts of H that do not really affect $\frac{x^T L_G x}{x^T L_H x}$ in the case when H is a subgraph of G.

Definition 3.3. *We say that a set of edges, S, is maximal if it is transitive when viewing it as a relation.*

3.1 Results for general graphs

Our first result allows us to consider arbitrary subgraphs of a graph G by expanding the vectors we are considering to match the vertices of G.

Lemma 3.4. *Let G be a graph and $H \subseteq G$, then $G \succeq H$*

Proof. By [2], we already know that the claim holds when $V_H = V_G$. If $V_H \neq V_G$, then we can simply consider the graph H' that that has the same edges as H, but vertex set being the same as G. Then, this expanded graph has the same laplacian quadratic form as H, being that no new edges were introduced, and has the same vertex set as G and so $G \succeq H$. ☐

Our next result gives a simple upper bound of the condition number between a graph and a subgraph, H, of it that can be strengthened under certain conditions when we consider C(H).

Lemma 3.5. *Let G be a graph, $S \subseteq E_G$, and $H = G - S$ having the same components as G. If J is a set of connected subgraphs of H satisfying $\forall (u,v) \in S \exists K \in J, u, v \in V_K$ and $H' = \cup_{K \in J} K \subseteq H$, then*

$$1 \leq \kappa(G,H) \leq 1 + \max_{\substack{x \in R^{V_{H'}} \\ \exists (u,v) \in S \\ x(u) \neq x(v)}} \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{K \in J} 2\mathcal{E}_K(x)} < \infty$$

Furthermore, if $J = C(H)$ and some maximizing vector satisfies $\sum_{(u,v)\notin S u,v \in V(S)}(x(u)-x(v))^2 = 0$ and for each component of $G - S$ having edges to only vertices of V(S) that have no edge in S we have $x(u) = x(v)$ for each u,v in V(S) with edges to this component in G, then second inequality is an equality. Moreover, if S is maximal, then the equality holds.

Proof. If $S = \varnothing$, then $G = H$ and $\kappa(G, H) = 1$. Consequently since the expression on the right is 1 + a non-negative number, we have that $\kappa(G, H)$ is at most this expression. On the other hand, suppose $S \neq \varnothing$. We note that since G and H have the same components, $N(L_G) = N(L_H)$. Now, $\forall x \notin N(L_G)$, with $x(u) \neq x(v)$ for some $(u, v) \in S$,

$$\begin{aligned}
\frac{x^T L_G x}{x^T L_H x} &= \frac{\sum_{(w,y)\in E_G}(x(w)-x(y))^2}{\sum_{(w,y)\in E_H}(x(w)-w(y))^2} && \text{Property of the Laplacian} \\
&= \frac{\sum_{(w,y)\in E_H}(x(w)-x(y))^2 + \sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{(w,y)\in E_H}(x(w)-w(y))^2} && \text{By definition of H} \\
&= 1 + \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{(w,y)\in E_H}(x(w)-x(y))^2} && \text{Since } x \notin N(L_G) = N(L_H) \\
&\leq 1 + \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{(w,y)\in E_{H'}}(x(w)-x(y))^2} && \text{By Lemma 3.3 since } H' \subseteq H \\
&= 1 + \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{K\in J}\sum_{(w,y)\in E_K}(x(w)-x(y))^2} && \text{By Definition of } H' \\
&= 1 + \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{K\in J} 2\mathcal{E}_K(x)} && \text{By definition of Energy}
\end{aligned}$$

Since $x(u) \neq x(v)$, we have by definition of J that there is some $K \in J$ satisfying $u, v \in V_K$ and K is connected implying it must contain a u-v path. Now, this u-v path must have at least one non-zero edge since its endpoints are not the same. Hence, the denominator of the last expression is not 0, so is defined. In addition, only the values of vertices in H' are used in the last expression, and hence the last expression need only consider $x \in R^{V_{H'}}$.

Now, for the case when $x(u) = x(v)$ for every $(u, v) \in S$, we have that $\frac{x^T L_G x}{x^T L_H x} = 1$, which we know is not the maximum since this would imply G is isomorphic to H which is impossible since $S \neq \varnothing$. Hence, if $x(u) = x(v)$ or every $(u, v) \in S$, we know that the vector cannot be a maximizer for the original expression. Thus, for any vector that could maximize $\frac{x^T L_G x}{x^T L_H x}$ we have that the last expression derived is defined and is larger, so the max over the last expression is larger. In other words,

$$\kappa(G, H) = \max\{1, \max_{x\notin N(L_G)} \frac{x^T L_G x}{x^T L_H x}\} = \max_{x\notin N(L_G)} \frac{x^T L_G x}{x^T L_H x} \leq 1 + \max_{\substack{x\in R^{V_{H'}} \\ \exists (u,v)\in S \\ x(u)\neq x(v)}} \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{K\in J} 2\mathcal{E}_K(x)}$$

In fact, the inequality can be made into an equality when $J = C(H)$ and the conditions on a maximizing vector hold. Specifically, suppose that $x \in R^{V_{CS(H)}}$ with $x(u) \neq x(v)$ for some $(u, v) \in S$ maximizes the second expression and satisfies the given conditions. We show how to construct a vector y from x that gives the same value in $\frac{y^T L_G y}{y^T L_H y}$. In particular, we show how to set the values of y so that the denominator becomes exactly $x^T L_{CS} x$ and since the two expressions have the same numerators this will complete the proof.

- First, we know that any vertex in a different component than a component containing some $u \in V(S)$ does not affect the second expression, so we remove them from the original expression by setting $y(w) = 0$ for each such vertex, w.
- Also, if w is a vertex in a component of $G - S$ having edges to only one vertex $t \in V(S)$, then we set $y(w) = x(t)$. This zeros out all the edges of that section of the component containing t.
- Similarly, if w is a vertex in some component of $G - S$ that has edges to vertices of V(S) that do not have corresponding edges in S, then we set $y(w) = x(t)$ where t is one such vertex of V(S) with edges

to the component. By assumption, each of these vertices have the same x value, and so this component is zeroed out.

- The remaining vertices not assigned a value are exactly the vertices belonging to $CS(H)$ and we set $y(w) = x(w)$ for any $w \in CS(H)$.

Consequently, the only remaining edges that differ between the two expressions are the edges between vertices of $V(S)$ that are not in S. However, by assumption x satisfies $\sum_{(u,v)\notin S u,v \in V(S)}(x(u)-x(v))^2 = 0$, so these edges are all zeroed out. Hence, we have that $\frac{y^T L_G y}{y^T L_H y} = 1 + \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{K\in C} 2\mathcal{E}_K(x)}$, so the maximum value of the second expression is achieved by the first. Hence, equality holds.

Now, for the moreover, we note that if S is a maximal set of edges, then any maximizing vector satisfies $\sum_{(u,v)\notin S u,v\in V(S)}(x(u)-x(v))^2 = 0$ since the sum is over the empty set. Also, if S is maximal then there are no components of $G - S$ having edges to vertices of V(S) that have no edges in S, so that claim is vacuously true. Hence, we may apply the furthermore to achieve equality. ☐

Our next lemma is merely a restating of a result from calculus that allows us to split a max into two maxes that partitions the vectors into vectors of smaller dimension. In particular, it states that when we fix a certain set of vertices and maximize an expression over the remaining vertices, then maximize the resulting expression, we end up with the maximum over all the vertices assuming the maximum of the non-fixed vertex set is finite.

Lemma 3.6. *Suppose G is a graph and $S \subset E_{\overline{G}}$. If* $\min_{x\in R^{V\setminus V(S)}} \sum_{(s,t)\in S}(x(s)-x(t))^2 > 0$ $\forall x \in R^{V(S)}$ *with $x(u) \neq x(v)$ for some $(u,v) \in S$, then*

$$\max_{\substack{x\in R^V \\ \exists (u,v)\in S \\ x(u)\neq x(v)}} \frac{\sum_{(s,t)\in S}(x(s)-x(t))^2}{\sum_{(s,t)\in E_G}(x(s)-x(t))^2} = \max_{\substack{x\in R^{V(S)} \\ \exists (u,v)\in S \\ x(u)\neq x(v)}} \frac{\sum_{(s,t)\in S}(x(s)-x(t))^2}{\min_{x\in R^{V\setminus V(S)}} \sum_{(s,t)\in E_G}(x(s)-x(t))^2}$$

Proof. First note, that for any non-negative function f, $\max_x \frac{a}{f(x)} = \frac{a}{\min_x f(x)}$. Hence, we have that when each $u \in V(S)$ is held constant,

$$\max_{x\in R^{V\setminus V(S)}} \frac{\sum_{(s,t)\in S}(x(s)-x(t))^2}{\sum_{(s,t)\in E_G}(x(s)-x(t))^2} = \frac{\sum_{(s,t)\in S}(x(s)-x(t))^2}{\min_{x\in R^{V\setminus V(S)}} \sum_{(s,t)\in E_G}(x(s)-x(t))^2}$$

Thus, we just need to show that the first expression in this equality is equal to our original expression. But, from Calculus, we know that if the max restricted over certain variables exists, then we can maximize the entire expression by maximizing over the other variables of the partially maximized function. Hence, since the min always exists, and so the partial max always exists we have that the claim holds. ☐

Lemma 3.7. *Let G be a graph, $S \subseteq E_G$, and $H = G - S$ having the same components as G. If J is a $V(S)$-internally disjoint set of connected subgraphs of H satisfying $\forall (u,v) \in S \exists K \in J, u,v \in V_K$ and $H' = \cup_{K\in J} K \subseteq H$ and $\forall K \in J \forall x \in R^{V(S)\cap V_K}$ with $x(u) \neq x(v)$ for some $(u,v) \in S, \min_{x\in R^{V_K\setminus V(S)}} 2\mathcal{E}_K(x) > 0$, then*

$$1 \le \kappa(G,H) \le 1 + \max_{\substack{x\in R^{V(S)} \\ \exists (u,v)\in S \\ x(u)\neq x(v)}} \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{K\in J} \min_{x\in R^{V_K\setminus V(S)}} 2\mathcal{E}_K(x)} < \infty$$

Furthermore, if $J = C$ and some maximizing vector satisfies $\sum_{(u,v)\notin S u,v\in V(S)}(x(u)-x(v))^2 = 0$ and for each component of $G - S$ having edges to only vertices of V(S) that have no edge in S we have $x(u) = x(v)$ for each u,v in V(S) with edges to this component in G, then the second inequality is an equality. Moreover, if S is maximal, then the equality holds.

Proof. This is an application of Lemma 3.5 and then Lemma 3.6 to H' and S. We note that the min can be pulled into the first sum being that the variables over which the inner sums are defined are different when the vertices incident to S are held fixed being that the subgraphs are all $V(S)$-internally disjoint. $\square$

3.2 Results for specific graphs

We introduce the notation $[n] = \{0, ..., n\}$ and $[n]^+ = \{1, ..., n\}$.

Lemma 3.8. *Suppose $u = p_0, p_1, ..., p_\ell = v$ is a path of length ℓ. If $\forall i \in [\ell - 1]^+, x(p_i) = \frac{x(p_{i-1})+x(p_{i+1})}{2}$, then $\forall i \in [\ell - 1]^+, x(p_i) = \frac{(\ell-i)x(p_{i-1})+x(v)}{\ell-i+1}$.*

Proof by Induction on i.

- **Basis:** if $i = \ell - 1$, then we have $x(p_i) = x(p_{\ell-1}) = \frac{x(p_{\ell-2})+x(p_\ell)}{2} = \frac{(\ell-i)x(p_{i-1})+x(v)}{\ell-i+1}$
- **Inductive Step:** Let $i > 0$ and suppose $\forall i < j \leq \ell - 1, x(p_j) = \frac{(\ell-j)x(p_{j-1})+x(v)}{\ell-j+1}$. We have

$$\begin{aligned}
x(p_i) &= \frac{x(p_{i-1}) + x(p_{i+1})}{2} && \text{By assumption} \\
&= \frac{x(p_{i-1}) + \frac{(\ell-(i+1))x(p_{(i+1)-1})+x(v)}{\ell-(i+1)+1}}{2} && \text{By the Induction Hypothesis} \\
&= \frac{(\ell-i)x(p_{i-1}) + (\ell-i+1)x(p_i) + x(v)}{2(\ell-i)} \\
\implies (1 - \frac{\ell-i+1}{2(\ell-i)})x(p_i) &= \frac{2(\ell-i) - (\ell-i+1)}{2(\ell-i)}x(p_i) = \frac{\ell-i+1}{2(\ell-i)}x(p_i) \\
&= \frac{(\ell-i)x(p_{i-1}) + x(v)}{2(\ell-i)} \\
\implies x(p_i) &= \frac{(\ell-i)x(p_{i-1}) + x(v)}{\ell-i+1}
\end{aligned}$$

$\square$

Lemma 3.9. *Suppose $u = p_0, p_1, ..., p_\ell = v$ is a path of length ℓ. If $\forall i \in [\ell - 1]^+, x(p_i) = \frac{x(p_{i-1})+x(p_{i+1})}{2}$, then $\forall i \in [\ell], x(p_i) = \frac{(\ell-i)x(u)+ix(v)}{\ell}$.*

Proof by induction on i.

- **Basis:** if i = 0, then $x(p_i) = x(p_0) = x(u) = \frac{(\ell-\ell)x(u)-0x(v)}{\ell} = \frac{(\ell-i)x(u)-ix(v)}{\ell}$
- **Inductive Step:** Let $0 < i \leq \ell$ and suppose $\forall 0 \leq j < i, x(p_j) = \frac{(\ell-j)x(u)+jx(v)}{\ell}$
  - if $i = \ell$, then $x(p_i) = x(v) = \frac{(\ell-\ell)x(u)+\ell x(v)}{\ell} = \frac{(\ell-i)x(u)+ix(v)}{\ell}$

– if $i < \ell$, then

$$
\begin{aligned}
x(p_i) &= \frac{(\ell - i)x(p_{i-1}) + x(v)}{\ell - i + 1} && \text{By Lemma 3.8} \\
&= \frac{(\ell - i)\frac{(\ell-(i-1))x(u)+(i-1)x(v)}{\ell} + x(v)}{\ell - i + 1} && \text{By the Induction Hypothesis} \\
&= \frac{\frac{(\ell-i)(\ell-i+1)x(u)+((\ell-i)(i-1)+\ell)x(v)}{\ell}}{\ell - i + 1} \\
&= \frac{(\ell - i)(\ell - i + 1)x(u) + i(\ell - i + 1)x(v)}{\ell(\ell - i + 1)} && (\ell - i)(i - 1) + \ell = i(\ell - i + 1) \\
&= \frac{(\ell - i)x(u) + ix(v)}{\ell}
\end{aligned}
$$

□

Lemma 3.10. *Suppose $u = p_0, p_1, ..., p_\ell = v$ is a path of length ℓ. Then, if we fix $x(u), x(v)$, we have $\min_{x \in R^{V \setminus \{u,v\}}} 2\mathcal{E}_{P_\ell}(x) = \frac{(x(u)-x(v))^2}{\ell}$*

Proof. We know that the minimizer of the energy when $x(u), x(v)$ are fixed is given by the function that is harmonic for the corresponding spring network where the fix set is $F = \{u, v\}$. Hence, the solution gives each vertices' value to be the degree weighted average of its neighbors. Since we are considering a path, we have $\forall i \in [\ell - 1]^+, x(p_i) = \frac{x(p_{i-1})+x(p_{i+1})}{2}$. Thus, by Lemma 3.9, we know that $\forall i \in \{0, ..., \ell\}, x(p_i) = \frac{(\ell-i)x(u)+ix(v)}{\ell}$. Thus,

$$
\begin{aligned}
\sum_{i=1}^{\ell}((x(p_{i-1}) - x(p_i))^2 &= \sum_{i=1}^{\ell}(\frac{(\ell - (i - 1))x(u) + (i - 1)x(v)}{\ell} - \frac{(\ell - i)x(u) + ix(v)}{\ell})^2 \\
&= \sum_{i=1}^{\ell}(\frac{(\ell - i + 1 - \ell + i)x(u) + (i - 1 - i)x(v)}{\ell})^2 \\
&= \sum_{i=1}^{\ell}(\frac{x(u) - x(v)}{\ell})^2 \\
&= \frac{1}{\ell^2}\sum_{i=1}^{\ell}(x(u) - x(v))^2 \\
&= \frac{\ell}{\ell^2}(x(u) - x(v))^2 = \frac{(x(u) - x(v))^2}{\ell}
\end{aligned}
$$

□

Lemma 3.11. *Let S be a set and f, x be functions defined on the elements of S. Then,*

$$
\frac{1}{(\sum_{u \in S} f(u))^2}\sum_{u \in S} f(u)(\sum_{v \in S \setminus \{u\}} f(v)x(u) - \sum_{v \in S \setminus \{u\}} f(v)x(v))^2 = \frac{1}{\sum_{u \in S} f(u)}\sum_{\{u,v\} \subseteq S} f(u)f(v)(x(u) - x(v))^2
$$

Proof. We first note that both expressions are multivariate polynomials over $x(S)$ where each variable has exponent at most 2. Hence, we prove the claim by showing for each variable $x(u)$, the coefficient of $x(u)$ and $x(u)^2$ are the same in both expressions. We start by categorizing the coefficients of the first expression.

- for $x(u)^2$, there are two cases to consider:

- When we consider the section of the outer sum defined over element u, we see the only way of forming $x(u)^2$ is with the $(\sum_{v\in S\setminus\{u\}} f(v)x(u))^2$ term that appears when expanding the square. This $= (\sum_{v\in S\setminus\{u\}} f(v))^2 x(u)^2$ giving rise to total coefficient $\frac{1}{(\sum_{u\in S} f(u))^2} f(u)(\sum_{v\in S\setminus\{u\}} f(v))^2$ in this case.
- Now, suppose we consider the section of the outer sum defined over some element $v \neq u$. In this case, the only way to get $x(u)^2$ is with the $(-f(u)x(u))^2$ term that appears when expanding the square. This gives rise to total coefficient $\frac{1}{(\sum_{u\in S} f(u))^2} f(v)f(u)^2$ in this case.

Hence, if we some over each coefficient that arises from each element over which the sum is defined, we get the total coefficient of $x(u)^2$ in the first expression is $\frac{1}{(\sum_{u\in S} f(u))^2}(f(u)(\sum_{v\in S\setminus\{u\}} f(v))^2 + \sum_{v\in S\setminus\{u\}} f(v)f(u)^2) = \frac{f(u)\sum_{v\in S\setminus\{u\}} f(v)}{(\sum_{u\in S} f(u))^2}(\sum_{v\in S\setminus\{u\}} f(v) + f(u)) = \frac{f(u)\sum_{v\in S\setminus\{u\}} f(v)}{(\sum_{u\in S} f(u))^2}\sum_{u\in S} f(u) = \frac{f(u)\sum_{v\in S\setminus\{u\}} f(v)}{\sum_{u\in S} f(u)}$.

- for $x(u)$ there are three cases to consider:
  - When we consider the section of the outer sum defined over element u we see the only way of forming $x(u)$ is by multiplying the term $\sum_{v\in S\setminus\{u\}} f(v)x(u)$ with a term of form $-f(v)x(v)$ for some $v \neq u$ in S that appears when expanding the square. Hence, the total coefficient's numerator in this case is $-2f(u)(\sum_{v\in S\setminus\{u\}} f(v))(\sum_{v\in S\setminus\{u\}} f(v)x(v))$. Note, it will be convenient to use the distributive law to get $= -2f(u)\sum_{v\in S\setminus\{u\}} f(v)\sum_{w\in S\setminus\{u\}} f(w)x(w)$
  $= -2f(u)(\sum_{v\in S\setminus\{u\}} f(v)\sum_{w\in S\setminus\{u,v\}} f(w)x(w) + \sum_{v\in S\setminus\{u\}} f(v)^2 x(v))$ by pulling v out of the inner sum and then out of the outer sum.
  - Now, consider the section of the outer sum defined over some element $v \neq u$ and consider only the negative coefficients that may arise. In particular, any such term must result from multiplying the term $-f(u)x(u)$ with the term $\sum_{w\in S\setminus\{v\}} f(w)x(v)$ that appears when expanding the square. Thus, the total numerator of the coefficient in this case is $-2f(v)f(u)\sum_{w\in S\setminus\{v\}} f(w)x(v)$.
  - Now we consider the positive coefficients for such v. These coefficients are all formed by multiplying the term $-f(u)x(u)$ with a term of the form $-f(w)x(w)$ for some $w \in S \setminus \{u,v\}$ that appears when expanding the square. Hence, the total numerator of the coefficient in this case is $2f(v)f(u)\sum_{w\in S\setminus\{u,v\}} f(w)x(w)$.

Thus, if we sum over all of the numerators for each element of the outer sum, we get the total numerator coefficient is

$$
\begin{aligned}
&-2f(u)(\sum_{v\in S\setminus\{u\}} f(v)\sum_{w\in S\setminus\{u,v\}} f(w)x(w) + \sum_{v\in S\setminus\{u\}} f(v)^2 x(v)) - 2f(u)\sum_{v\in S\setminus\{u\}} f(v)\sum_{w\in S\setminus\{v\}} f(w)x(v) \\
&+ 2f(u)\sum_{v\in S\setminus\{u\}} f(v)\sum_{w\in S\setminus\{u,v\}} f(w)x(w) \\
&= -2f(u)(\sum_{v\in S\setminus\{u\}} f(v)^2 x(v) + \sum_{v\in S\setminus\{u\}} f(v)\sum_{w\in S\setminus\{v\}} f(w)x(v)) \\
&= -2f(u)\sum_{v\in S\setminus\{u\}} f(v)x(v)(f(v) + \sum_{w\in S\setminus\{v\}} f(w)) \\
&= -2f(u)(\sum_{w\in S} f(w))(\sum_{v\in S\setminus\{u\}} f(v)x(v))
\end{aligned}
$$

Hence, the total coefficient is $\frac{-2f(u)(\sum_{w\in S} f(w))(\sum_{v\in S\setminus\{u\}} f(v)x(v))}{(\sum_{w\in S} f(w))^2} = \frac{-2f(u)\sum_{v\in S\setminus\{u\}} f(v)x(v)}{\sum_{w\in S} f(w)}$

Now, we have that

$$
\begin{aligned}
&\frac{1}{\sum_{u\in S} f(u)} \sum_{\{u,v\}\subseteq S} f(u)f(v)(x(u)-x(v))^2 \\
&= \frac{1}{2\sum_{u\in S} f(u)} \sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)(x(u)-x(v))^2 \\
&= \frac{1}{2\sum_{u\in S} f(u)} \sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)(x(u)^2+x(v)^2-2x(u)x(v)) \\
&= \frac{1}{2\sum_{u\in S} f(u)} (\sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)x(u)^2 + \sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)x(v)^2 - 2\sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)x(u)x(v)) \\
&= \frac{1}{2\sum_{u\in S} f(u)} (2\sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)x(u)^2 - 2\sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)x(u)x(v)) \\
&= \frac{1}{\sum_{u\in S} f(u)} (\sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)x(u)^2 - \sum_{u\in S}\sum_{v\in S\setminus\{u\}} f(u)f(v)x(u)x(v)) \\
&= \frac{\sum_{u\in S} f(u)(\sum_{v\in S\setminus\{u\}} f(v))x(u)^2 - 2\sum_{\{u,v\}\subseteq S} f(u)f(v)x(u)x(v)}{\sum_{u\in S} f(u)}
\end{aligned}
$$

From this alternate representation, it is easily seen that for any $u \in S$, the coefficient of $x(u)^2$ is exactly the one derived previously and similarly for $x(u)$. Hence, the two expressions are equal. □

Lemma 3.12. *Let $S \subseteq E_{K_\ell}$. If $x(V(S))$ are fixed, then* $\min_{x\in R^{V\setminus V(S)}} 2\mathcal{E}_{K_\ell - S}(x) = \frac{\ell-|V(S)|}{|V(S)|}\sum_{\{u,v\}\subseteq V(S)}(x(u)-x(v))^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}} (x(u)-x(v))^2$

Proof. Again, the minimizer is exactly the function that is harmonic for the spring network where the fixed set $F = S$. Hence, we have that each vertices' value is the degree weighted average of its neighbors. That is, $\forall w \in V \setminus V(S), x(w) = \frac{1}{\ell-1}\sum_{y\in V\setminus\{w\}} x(y)$. Now, for any two vertices, w and y, not in $V(S)$, consider $x(w) - x(y)$:

$$
\begin{aligned}
x(w) - x(y) &= \frac{\sum_{s\in V\setminus\{w\}} x(s) - \sum_{t\in V\setminus\{y\}} x(t)}{\ell - 1} \\
&= \frac{x(y) - x(w)}{\ell - 1} \\
&\implies (x(w) - x(y)) + \frac{x(w) - x(y)}{\ell - 1} = 0 \\
&\implies \frac{\ell}{\ell - 1}(x(w) - x(y)) = 0 \\
&\implies x(w) = x(y)
\end{aligned}
$$

Hence, we have:

$$\begin{aligned}
x(w) &= \frac{1}{\ell-1}\sum_{s\in V\setminus\{w\}} x(s)\\
&= \frac{1}{\ell-1}(\sum_{s\in V\setminus\{w\}\cup V(S)} x(w) + \sum_{u\in V(S)} x(u))\\
&= \frac{\ell-1-|V(S)|}{\ell-1}x(w) + \frac{1}{\ell-1}(\sum_{u\in V(S)} x(u))\\
\implies \frac{|V(S)|}{\ell-1}x(w) &= \frac{1}{\ell-1}(\sum_{u\in V(S)} x(u))\\
\implies x(w) &= \frac{\sum_{u\in V(S)} x(u)}{|V(S)|}
\end{aligned}$$

Hence, if we plug in this minimizer into the sum we will get the minimum value.

$$\begin{aligned}
\sum_{(s,t)\in E_G}(x(s)-x(t))^2 &= \sum_{u\in V(S)}\sum_{w\notin V(S)}(x(u)-x(w))^2 + \sum_{w\in V\setminus V(S)}\sum_{y\in V\setminus V(S)\cup\{w\}}(x(w)-x(y))^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2\\
&= \sum_{u\in V(S)}(\ell-|V(S)|)(x(u)-x(w))^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2\\
&= \sum_{u\in V(S)}(\ell-|V(S)|)(x(u)-\frac{\sum_{v\in V(S)} x(v)}{|V(S)|})^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2\\
&= \frac{\ell-|V(S)|}{|V(S)|^2}\sum_{u\in V(S)}((|V(S)|-1)x(u)-\sum_{v\in V(S)\setminus\{u\}}x(v))^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2\\
&= \frac{\ell-|V(S)|}{|V(S)|}\sum_{\{u,v\}\subseteq V(S)}(x(u)-x(v))^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2
\end{aligned}$$

Where the last equality comes from applying Lemma 3.11 with $f=1$, $S=V(S)$, and x being the same x. ☐

Lemma 3.13. *If G is a connected graph, $S\subseteq E_{\overline{G}}$, $x(V(S))$ are fixed, and $d^S(u)=d(u)-|\{(u,v)\in E_G|v\in V(S)\}|$, then if $2\mathcal{E}_G(x)$ is minimized when $\forall w,y\in V\setminus V(S), x(w)=x(y)$, we have*

$$\min_{x\in R^{V\setminus V(S)}} 2\mathcal{E}_G(x) = \frac{\sum_{\{u,v\}\subseteq V(S)} d^S(u)d^S(v)(x(u)-x(v))^2}{\sum_{u\in V(S)} d^S(u)} + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2$$

Proof.

$$\begin{aligned}
\sum_{(s,t)\in E_G}(x(s)-x(t))^2 &= \sum_{\substack{(u,w)\in E_G\\ u\in V(S),w\notin V(S)}}(x(u)-x(w))^2 + \sum_{\substack{(s,t)\in E_G\\ s,t\notin V(S)}}(x(s)-x(t))^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2\\
&= \sum_{u\in V(S)} d^S(u)(x(u)-x(w))^2 + \sum_{\substack{(u,v)\notin S\\ u,v\in V(S)}}(x(u)-x(v))^2
\end{aligned}$$

Now, we will find the optimal value of $x(w)$ for each w not in V(S) in terms of the vertices of V(S).

$$\frac{\partial}{\partial x(w)} \sum_{u \in V(S)} d^S(u)(x(u) - x(w))^2 = 0$$
$$\implies \sum_{u \in V(S)} d^S(u)(x(w) - x(u)) = 0$$
$$\implies \sum_{u \in V(S)} d^S(u)x(w) = \sum_{u \in V(S)} d^S(u)x(u)$$
$$\implies x(w) = \frac{\sum_{u \in V(S)} d^S(u)x(u)}{\sum_{u \in V(S)} d^S(u)}$$

Hence, plugging in the minizimizer yields the minimum value. First, consider just the first sum.

$$\begin{aligned}\sum_{u \in V(S)} d^S(u)(x(u) - x(w))^2 &= \sum_{u \in V(S)} d^S(u)(x(u) - \frac{\sum_{v \in V(S)} d^S(v)x(v)}{\sum_{u \in V(S)} d^S(u)})^2 \\ &= \frac{1}{(\sum_{u \in V(S)} d^S(u))^2} \sum_{u \in V(S)} d^S(u)(\sum_{v \in V(S) \setminus \{u\}} d^S(v)x(u) - \sum_{v \in V(S) \setminus \{u\}} d^S(v)x(v))^2 \\ &= \frac{1}{\sum_{u \in V(S)} d^S(u)} \sum_{\{u,v\} \subseteq V(S)} d^S(u)d^S(v)(x(u) - x(v))^2\end{aligned}$$

Where the last equality follows from Lemma 3.11 with $f = d^S$, $S = V(S)$, and x being the same x. Hence, by adding back the sum on the right the claim holds. □

Corollary 3.13.1. *If $S = \{(u, v)\}$, then the minimum from Lemma 3.13 is $\geq \frac{1}{2}(x(u) - x(v))^2$.*

Proof. We know that $d(u), d(v) \geq 1$ since G is connected, so there is some $u - v$ path in G. Now, there are three cases to consider:

- If $d(u), d(v) \geq 2$, then $d(u)d(v) \geq d(u) + d(v)$. Thus, $\frac{d(u)d(v)}{d(u)+d(v)} \geq 1 \geq \frac{1}{2}$
- If without loss of generality $d(u) = 1$ and $d(v) > 1$, then $\frac{d(u)d(v)}{d(u)+d(v)} = \frac{d(v)}{1+d(v)}$. Now, $\min_{k \in Z^+} \frac{k}{k+1} = \frac{1}{2}$ since its an increasing function over the positive integers, so the minimum value results from plugging in the minimum value of the domain. Hence, $\frac{d(v)}{1+d(v)} \geq \frac{1}{2}$.
- If $d(u) = d(v) = 1$, then $\frac{d(u)d(v)}{d(u)+d(v)} = \frac{1 \times 1}{1+1} = \frac{1}{2}$

□

4 Upper Bounds

In this section we present several functions for which graphs G and H are spectrally isomorphic that are defined with respect to various combinatorial properties of G and H.

4.1 Results for general graphs

We begin with our main result that describes a function dependent on the subgraphs present in the graph H.

Theorem 4.1 (Subtracting Lemma)**.** *Let G be a graph, $S \subseteq E_G$, $H = G - S$ having the same components as G. Also, suppose $J = K \cup E \cup P$ is a set of $V(S)$-internally disjoint and edge disjoint subgraphs of H, where K contains subgraphs of the form $K_\ell - T$ for $\varnothing \neq T \subseteq S, \ell \geq 3$, E contains subgraphs excluding those in K containing u and v for some $(u, v) \in S$ and satisfying $2\mathcal{E}_L(x)$ for $x(V(S) \cap V_L)$ fixed is minimized when each non fixed vertex has equal x value, and P is a set of $u - v$ paths in H for $(u, v) \in S$ such that no internal node of the path is in $V(S)$. Furthermore, suppose that $\forall (u, v) \in S \exists L \in J, u, v \in V_L$. Then, $G \cong_s^\alpha H$ for*

$$\alpha = 1 + \max_{(u,v)\in S} \frac{1}{\sum_{\substack{L\in K\\u,v\in V_L}} \frac{|V_L|-|V_L\cap V(S)|}{|V_L\cap V(S)|} + \sum_{\substack{L\in E\\u,v\in V_L}} \frac{d_L^S(u)d_L^S(v)}{\sum_{w\in V_L\cap V(S)} d_L^S(w)} + \sum_{\substack{L\in P\\u,v\in V_L}} \frac{1}{|E_L|}}$$

Furthermore, $\kappa(G, H) = \alpha$ if $J = C(H)$ and there exists some maximizing $(u, v) \in S$ such that for every $w \in V(S) \setminus \{u, v\}$ in the same component of H as u and v, w can only reach v [resp. u] in H through u [resp. v] or through a vertex of $V(S)$ in a subgraph of C containing both u and v. Also, if w itself is in a subgraph of C containing both u and v, then w must not have any edge to a vertex of $V(S)$ that has a path to v [resp. u] either through edges between vertices of $V(S)$ or that is in a subgraph of C with vertices of $V(S)$ that can only reach u [resp. v] through v [resp. u] or through a vertex of $V(S)$ in a subgraph of C containing both u and v. Moreover, if S is maximal and each edge of S achieves the max defined in α then $\kappa(G, H) = \alpha$.

Proof. Consider the identity permutation. The edge disjointness criteria ensures that $\cup_{L\in J} L \subseteq H$, so we can apply Lemma 3.6 to get $\kappa(G, H) \leq 1 + \max_{\substack{x\in R^{V(S)}\\ \exists (u,v)\in S\\ x(u)\neq x(v)}} \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{L\in J}\min_{x\in R^{V_L\setminus V(S)}} 2\mathcal{E}_L(x)}$. Hence, we show a lower bound on the denominator to get an upper bound on the second expression.

$$= \sum_{L\in K} \min_{x\in R^{V_L\setminus V(S)}} 2\mathcal{E}_L(x) + \sum_{L\in E} \min_{x\in R^{V_L\setminus V(S)}} 2\mathcal{E}_L(x) + \sum_{L\in P} \min_{x\in R^{V_L\setminus V(S)}} 2\mathcal{E}_L(x)$$

$$= \sum_{L\in K} \frac{|V_L| - |V_L \cap V(S)|}{|V_L \cap V(S)|} \sum_{\{u,v\}\subseteq V(S)\cap V_L} (x(u) - x(v))^2 + \sum_{L\in E} \frac{\sum_{\{u,v\}\subseteq V(S)\cap V_L} d_L^S(u)d_L^S(v)(x(u) - x(v))^2}{\sum_{w\in V_L\cap V(S)} d_L^S(w)}$$

$$+ \sum_{\substack{L\in P\\u,v\in V_L\cap V(S)}} \frac{1}{|E_L|}(x(u) - x(v))^2$$

$$= \sum_{(u,v)\in S} \Big(\sum_{\substack{L\in K\\u,v\in V(S)}} \frac{|V_L| - |V_L \cap V(S)|}{|V_L \cap V(S)|}(x(u) - x(v))^2 + \sum_{\substack{L\in E\\u,v\in V(S)}} \frac{d_L^S(u)d_L^S(v)(x(u) - x(v))^2}{\sum_{w\in V_L\cap V(S)} d_L^S(w)} + \sum_{\substack{L\in P\\u,v\in V_L}} \frac{1}{|E_L|}(x(u) - x(v))^2\Big)$$

$$+ \sum_{\substack{(u,v)\notin S\\u,v\in V(S)}} a(u, v)(x(u) - x(v))^2$$

$$\geq \sum_{(u,v)\in S} \Big(\sum_{\substack{L\in K\\u,v\in V(S)}} \frac{|V_L| - |V_L \cap V(S)|}{|V_L \cap V(S)|} + \sum_{\substack{L\in E\\u,v\in V(S)}} \frac{d_L^S(u)d_L^S(v)}{\sum_{w\in V_L\cap V(S)} d_L^S(w)} + \sum_{\substack{L\in P\\u,v\in V_L}} \frac{1}{|E_L|}\Big)(x(u) - x(v))^2$$

$$\geq \min_{(u,v)\in S} \Big(\sum_{\substack{L\in K\\u,v\in V(S)}} \frac{|V_L| - |V_L \cap V(S)|}{|V_L \cap V(S)|} + \sum_{\substack{L\in E\\u,v\in V(S)}} \frac{d_L^S(u)d_L^S(v)}{\sum_{w\in V_L\cap V(S)} d_L^S(w)} + \sum_{\substack{L\in P\\u,v\in V_L}} \frac{1}{|E_L|}\Big) \sum_{(u,v)\in S} (x(u) - x(v))^2$$

Where the first equality comes from the fact that $J = K \cup E \cup P$. Then, the second equality comes from applying the optimization lemmas to the special types of graphs. Next, the third equality results from swapping the inner and outer sums and rearranging the terms based on the edges. By the assumption that every edge of S appears in at least one subgraph in J, we know that there is a positive term in the first sum for each edge of S. Now, the first inequality comes from removing the terms associated with edges not in S but are between vertices incident to S. Lastly, the second inequality comes from lower bounding each entry of the outer sum with the minimum term in the sum and then pulling the term out. Now, we can plug this expression back into the denominator of the max to get:

$$1 + \max_{\substack{x \in R^{V(S)} \\ \exists (u,v) \in S \\ x(u) \neq x(v)}} \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{L\in J} \min_{x \in R^{V_L \setminus V(S)}} 2\mathcal{E}_L(x)}$$

$$\leq 1 + \max_{\substack{x \in R^{V(S)} \\ \exists (u,v) \in S \\ x(u) \neq x(v)}} \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\min_{(u,v)\in S}\left(\sum_{\substack{L\in K \\ u,v\in V_L}} \frac{|V_L|-|V_L\cap V(S)|}{|V_L \cap V(S)|} + \sum_{\substack{L\in E \\ u,v\in V_L}} \frac{d_L^S(u)d_L^S(v)}{\sum_{w\in V_L\cap V(S)} d_L^S(w)} + \sum_{\substack{L\in P \\ u,v\in V_L}} \frac{1}{|E_L|}\right) \sum_{(u,v)\in S}(x(u)-x(v))^2}$$

$$= 1 + \frac{1}{\min_{(u,v)\in S}\left(\sum_{\substack{L\in K \\ u,v\in V_L}} \frac{|V_L|-|V_L\cap V(S)|}{|V_L \cap V(S)|} + \sum_{\substack{L\in E \\ u,v\in V_L}} \frac{d_L^S(u)d_L^S(v)}{\sum_{w\in V_L\cap V(S)} d_L^S(w)} + \sum_{\substack{L\in P \\ u,v\in V_L}} \frac{1}{|E_L|}\right)}$$

$$= 1 + \max_{(u,v)\in S} \frac{1}{\sum_{\substack{L\in K \\ u,v\in V_L}} \frac{|V_L|-|V_L\cap V(S)|}{|V_L \cap V(S)|} + \sum_{\substack{L\in E \\ u,v\in V_L}} \frac{d_L^S(u)d_L^S(v)}{\sum_{w\in V_L\cap V(S)} d_L^S(w)} + \sum_{\substack{L\in P \\ u,v\in V_L}} \frac{1}{|E_L|}}$$

Thus, we have that $\forall x \notin N(L_G)$

$$\frac{x^T L_G x}{x^T L_H x} \leq \max_{x \notin N(L_G)} \frac{x^T L_G x}{x^T L_H x} \leq 1 + \max_{\substack{x \in R^{V(S)} \\ \exists (u,v) \in S \\ x(u) \neq x(v)}} \frac{\sum_{(u,v)\in S}(x(u)-x(v))^2}{\sum_{L\in J} \min_{x \in R^{V_L \setminus V(S)}} 2\mathcal{E}_L(x)} \leq \alpha$$

Hence, $G \cong_s^\alpha H$.

Now, we argue that under the conditions of the furthermore, we can construct a vector x so that $\frac{x^T L_G x}{x^T L_H x} = \alpha$.

- First, we arbitrarily choose $x(u) \neq x(v)$ and set $x(w) = 0$ for any w not in the component of H containing u and v.
- Next, we set $x(w) = x(u)$ for any w in $G - \{u\}$ that does not contain any vertex of $V(S)$, and similarly do the same for v.
- Now, we eliminate all the edges between vertices of $V(S)$ that are not edges of S by setting $x(w) = x(u)$ if w is reachable from u using only edges between vertices of $V(S)$ that are present in H. We similarly do this for v. For any other such edge (w, h) we set $x(w) = x(h)$. Note, by the furthermore condition, there is never a path from u to v using only edges between vertices of $V(S)$ and so we can safely remove the edges this way without ever running into the problem of needing to set $x(u) = x(v)$.
- Next, we eliminate all edges that aren't in a subgraph of C containing both u and v by setting $x(w) = x(u)$ for such w that can only reach v through u or some vertex of $V(S)$ that is in a subgraph of C containing both u and v and set the value of that vertex to $x(u)$ as well. We similarly do this for v. Lastly, for any vertex that can only reach u and v by passing through some vertex, w, of $V(S)$ that is in a subgraph of C containing u and v where w cannot reach u or v using only edges between vertices of $V(S)$ and w is not adjacent to some subgraph of C, other than those containing u and v, that can reach u or v, then we set the vertex's value to 0 arbitrarily and set $x(w) = 0$. Note that the furthermore condition ensures it is safe to do zero out edges this way since the only possible conflict would arise if some w in a subgraph of C with both u and v had a path to both u and v using edges

of V(S), which cannot happen by assumption, or if it was contained in a component of C with vertices having paths to u and a component of C with vertices having paths to v. However, the furthermore condition explicitly guarantees that w may only be apart of components that can only reach v by going through u first or some other vertex in a subgraph with both u and v, and similarly if we interchange u and v.

- Now, the current vector constructed zeros out everything that is not a subgraph of C containing both u and v, and we can just set the remaining entries of the vector as described in the optimization lemmas to coincide with the minimizers of the energies of each of these subgraphs.

Thus, α can be achieved and so it is the maximum of $\frac{x^T L_G x}{x^T L_H x}$, which is the relative condition number in this case. On the other hand, for the moreover, we have that if S is maximal then Lemma 3.7 and the fact that there are no edges of form $(u, v) \notin S$ yet $u, v \in V(S)$ gives each inequality in the first sequence of results is an equality except for the last. However, since edge edge of S by assumption achieves the max, it must be each edge also achieves the same min and so the last inequality is also an equality and the claim holds. ☐

We can restrict this result to get some more interesting though looser upper bounds.

Corollary 4.1.1. *Let G be a graph, $S \subset E_G$, and suppose $H = G - S$ has the same components as G. Also, suppose P is a set of vertex disjoint paths in H where each path connects the endpoints of of an edge that is in S. Then, $G \cong_s^\alpha H$, where*

$$\alpha = 1 + \max_{L \in P} |E_L|$$

Proof. This follows immediately from Theorem 4.1 with $J = P$ since for each edge there is exactly one path and we know that $\frac{1}{\frac{1}{\frac{1}{|E_L|}}} = |E_L|$. ☐

Corollary 4.1.2. *Let G be a graph, $(u, v) \in E_G$, and $H = G - (u, v)$ having the same components as G. Also, let K be the set of all $L \in C(H)$ such that $L = K_\ell - (u, v)$ for some ℓ, E be the set of all $L \in C(H)$ excluding those in K such that $2\mathcal{E}_L(x)$ for $x(u), x(v)$ fixed is minimized when each non u,v vertex has equal x value, and P be the set of all other elements of C(H). Then, $G \cong_s^\alpha H$ for*

$$\alpha = 1 + \frac{1}{\sum_{L \in K} \frac{|V_L| - 2}{2} + \sum_{L \in E} \frac{d_L(u) d_L(v)}{d_L(u) + d_L(v)} + \sum_{L \in P} \frac{1}{d_L(u, v)}}$$

Furthermore, if every element of P is a u-v path, then $\kappa(G, H) = \alpha$.

Proof. This is a simple application of Theorem 4.1 by letting J be C(H) where we replace any subgraph in C(H) that is not of the first two forms with the u-v path that it must contain since each subgraph in C(H) is a connected subgraph containing both u and v. If each element of C(H) is already of one of the three forms then we let J be C(H) unaltered and so can apply the furthermore condition of the theorem since S has only one edge so satisfies the condition giving that the relative condition number is α. ☐

Corollary 4.1.3. *If G and $H = G - S$ for $S \subseteq E_G$ have the same components and there exists a non-empty set, K, of subgraphs of H having form $K_\ell - S$ for some $\ell > |V(S)|$, then $G \cong_s^{1+\frac{|V(S)|}{\max_{L \in K} |V_L| - |V(S)|}} H$. Moreover, if some maximum clique in G contains S, then $G \cong_s^{\frac{|V(S)|}{\omega(G) - |V(S)|}} H$.*

Proof. We apply Theorem 4.1 to $J = K$. Then, we upper bound the resulting *al* by noting that the sum in the denominator of the fractional part of α is lower bounded by any single element of the sum. In particular, the largest element. Then simplifying the expression gives the result. The moreover then easily follows by definition of the clique number. ☐

Corollary 4.1.4. *If G and $H = G - (u, v)$ for $(u, v) \in E_G$ have the same components, P is a maximum set of vertex disjoint u-v paths in H, and K is a set of $\{u, v\}$-internally disjoint subgraphs of H of form $K_\ell - (u, v), \ell \geq 3$, then*

1. $G \cong_s^{1+d_H(u,v)} H$
2. $G \cong_s^{n} H$
3. $G \cong_s^{1+\frac{\max_{L\in P}|E_L|}{\lambda_H(u,v)}} H$
4. $G \cong_s^{1+\frac{2}{|K|(\min_{L\in K}|V_L|-2)}} H$

Proof. Each result holds by merely applying Theorem 4.1 to a specific set of subgraphs of H and then upper bounding the α given by the theorem if necessary.

1. This follows from Theorem 4.1 by letting J to be the singleton set containing the shortest u-v path in H.
2. This follows from 1. since the shortest u-v path can be at most $n - 1$ and there exists a u-v path since G and H have the same components.
3. By definition, there are $\lambda_H(u, v)$ many paths in P. Now, we have that the denominator of the fractional part of the α that is given by Theorem 4.1 when we let $J = P$ is lower bounded by $\lambda_H(u, v) \min_{L\in P} \frac{1}{|E_L|} = \frac{\lambda_H(u,v)}{\max_{L\in P}|E_L|}$. Hence, the claim holds.
4. By definition, there are $|K|$ many subgraphs in K. Now, we have that the denominator of the fractional part of the α that is given by Theorem 4.1 when we let $J = K$ is lower bounded by $|K| \min_{L\in K} \frac{|V_L|-2}{2} = \frac{|K|(min_{L\in K}|V_L|-2)}{2}$. Hence, the claim holds.

□

We can also always show to graphs are spectrally isomorphic by iteratively removing or adding edges and applying transitivity of the relation.

Lemma 4.2 (Swapping Lemma)**.** *Let G and H be graphs differing by k edges. Then, if we choose any ordering of edge additions and deletions, $(e_1, e_2, ..., e_k)$, so that $G = G_0$, $G_k = H$, and for each i either $G_i = G_{i-1} - e_i$ or $G_i = G_{i-1} + e_i$ and $G_{i-1} \cong_s^{\alpha} G_i$, then $G \cong_s^{\alpha} H$ where*

$$\alpha = \prod_{i=1}^{k} \alpha_i$$

Proof by induction on k.

- **Basis:** If $k = 0$, then $G = H$ and $\alpha = \prod_{i=1}^{0} \alpha_i = 1$ and we know that $G \cong_s^1 G = H$
- **Inductive Step:** If $k > 0$, as above fix an ordering of the edges to be added to G and deleted from G and suppose inductively that $G \cong_s^{\beta} G_{k-1}$ for $\beta = \prod_{i=1}^{k-1} \alpha_i$. Then, we know by assumption on the ordering that $G_{k-1} \cong_s^{\alpha_k} G_k = H$. Hence, the Induction Hypothesis and transitivity of $\cong_s$ gives that $G \cong_s^{\gamma} H$ where $\gamma = \beta\alpha_k = (\prod_{i=1}^{k-1} \alpha_i)\alpha_k = \alpha$.

□

Lemma 4.3. *Let G and H be graphs with the same components and that differ by k edges. Then, if we choose any ordering of edge additions and deletions, $(e_1, e_2, ..., e_k)$, so that $G = G_0$, $G_k = H$, and for each i either $G_i = G_{i-1} - e_i$ or $G_i = G_{i-1} + e_i$ and G_{i-1} has the same components as G_i, then for each i $G_{i-1} \cong_s^{\alpha_i} G_i$ where α_i is the term given in Corollary 4.1.2 and $G \cong_s^{\alpha} H$ where*

$$\alpha = \prod_{i=1}^{k} \alpha_i$$

Proof. There are two cases to consider. If $G_i = G_{i-1} - e_i$, then since G_{i-1} and G_i have the same components by assumption, we have by Theorem 4.1 that $G_i \cong_s^{\alpha_i} G_{i-1}$ where α_i is the bound given by the same theorem. Hence, by symmetry of $\cong_s$, we know that $G_{i-1} \cong_s^{\alpha_i} G_i$. Alternatively, if $G_i = G_{i-1} + e_i$, then $G_{i-1} = G_i - e_i$, so again the assumption allows us to apply Theorem 4.1 to get that $G_{i-1} \cong_s^{\alpha_i} G_i$ with α_i. Now, we can simply apply the swapping lemma to conclude that $G \cong_s^{\alpha} H$. □

Proposition 4.4. *Let G and H be graphs with the same components and that differ by k edges. Then, there exists an ordering of edge additions and deletions, $(e_1, e_2, ..., e_k)$, so that $G = G_0$, $G_k = H$, and for each i either $G_i = G_{i-1} - e_i$ or $G_i = G_{i-1} + e_i$ and G_{i-1} has the same components as G_i.*

Proof. Since G and H have the same components, we can partition the edge additions and deletions to additions and deletions on each component since any edge added between two components would need to be removed anyway. Hence, we can just consider connected graphs G and H. For G and H connected, we can always just perform all of the edge additions first, which cannot change the component since adding edges to a connected graph cannot form new components. Then, we can just perform all of the edge deletions. We know no edge deletion can disconnect the graph since if it did then the remaining edge deletions could not somehow reconnect the graph in order to get H which is connected. Thus, arbitrarily choosing an ordering of additions and then an ordering of subtractions and concatenating the two orderings together gives an ordering of edge additions and deletions that satisfies the claim. Now, for disconnected graphs, we just construct the sequences for each component and arbitrarily concatenate them together. □

Proposition 4.5. *If G and H have the same component structure and G has k components, then $G \cong_s^{\alpha} H$ for $\alpha = \max\{\frac{\lambda_n(G)}{\lambda_{k+1}(H)}, \frac{\lambda_n(H)}{\lambda_{k+1}(G)}\}$. Furthermore, if $H \subseteq G$, then the claim holds for $\alpha = \frac{\lambda_n(G)}{\lambda_{k+1}(H)}$*

Proof. Let π be the permutation constructed in the proof of Lemma 2.1 satisfying that G and $\pi(H)$ have the same components. Then,

$$\max_x \frac{x^T L_G x}{x^T L_{\pi(H)} x} = \max_x \frac{x^T L_G x}{x^T L_{\pi(H)} x} \times \frac{\frac{1}{x^T x}}{\frac{1}{x^T x}} = \max_x \frac{\frac{x^T L_G x}{x^T x}}{\frac{x^T L_{\pi(H)}}{x^T x}} \leq \frac{\max_x \frac{x^T L_G x}{x^T x}}{\min_x \frac{x^T L_{\pi(H)} x}{x^T x}} = \frac{\lambda_n(G)}{\lambda_{k+1}(H)} \leq \max\{\frac{\lambda_n(G)}{\lambda_{k+1}(H)}, \frac{\lambda_n(H)}{\lambda_{k+1}(G)}\}$$

Where the last equality holds since permuting a matrix does not change the eigenvalues, i.e $\lambda_{k+1}(\pi(H)) = \lambda_{k+1}(H)$. Also, we have

$$\min_x \frac{x^T L_G x}{x^T L_{\pi(H)} x} = \min_x \frac{x^T L_G x}{x^T L_{\pi(H)} x} \times \frac{\frac{1}{x^T x}}{\frac{1}{x^T x}} = \min_x \frac{\frac{x^T L_G x}{x^T x}}{\frac{x^T L_{\pi(H)}}{x^T x}} \geq \frac{\min_x \frac{x^T L_G x}{x^T x}}{\max_x \frac{x^T L_{\pi(H)} x}{x^T x}} = \frac{\lambda_{k+1}(G)}{\lambda_n(H)} \geq \min\{\frac{\lambda_{k+1}(G)}{\lambda_n(H)}, \frac{\lambda_{k+1}(H)}{\lambda_n(G)}\}$$

Finally, we have $\min\{\frac{\lambda_{k+1}(G)}{\lambda_n(H)}, \frac{\lambda_{k+1}(H)}{\lambda_n(G)}\} = \frac{1}{\max\{\frac{\lambda_n(G)}{\lambda_{k+1}(H)}, \frac{\lambda_n(H)}{\lambda_{k+1}(G)}\}}$. Hence, since the bounds hold for the extremes, they hold for all x not in the null space of the laplacian of G. Now, if $H \subseteq G$, then we already know $\frac{x^T L_G x}{x^T L_H x} \geq 1$ for all x not in the null space of the laplacian of G, and so we only need the upper bound $\frac{\lambda_n(G)}{\lambda_{k+1}(H)}$. □

Lemma 4.6. *Suppose $G = G_1 + G_2 + ... + G_k$ and $H = H_1 + H_2 + ... + H_k$ are graphs with k components, and $G \cong_s^\alpha H$. If $\exists\pi$ showing that $G \cong_s^\alpha H$ so that for each i $V_{\pi(H_j)} = V_{G_i}$ and $\pi(H_j) = f(H_i)$ for some permutation f, then $\forall i, G_i \cong_s^\alpha H_i$.*

Proof by Contrapositive. Suppose there is some i such that $G_i \ncong_s^\alpha H_i$. Then, we have by definition that $\forall\pi'\exists x \notin N(L_{G_i})$ such that either $\frac{x^T L_{G_i} x}{x^T L_{\pi'(H_i)} x} > \alpha$ or $\frac{x^T L_{G_i} x}{x^T L_{\pi'(H_i)} x} < \frac{1}{\alpha}$. Also, let π be any permutation of H. If $V_{\pi(H_j)} \neq V_{G_i}$ or $\pi(H_j) \neq f(H_i)$ for some permutation f, we are done. Otherwise, suppose $V_{\pi(H_j)} = V_{G_i}$ and $\pi(H_j) = f(H_i)$ for some permutation f. Consider the vector x that is the same x as previously defined over the vertices of G_i but is 0 in each other entry. Then, $\frac{x^T L_G x}{x^T L_{\pi(H)} x} = \frac{x^T L_{G_i} x}{x^T L_{\pi(H_j)} x} = \frac{x^T L_{G_i} x}{x^T L_{f(H_i)} x}$. However, this means that $\frac{x^T L_G x}{x^T L_{\pi(H)} x} > \alpha$ or $\frac{x^T L_G x}{x^T L_{\pi(H)} x} < \frac{1}{\alpha}$ for this x which is not in the null space of L_G since its not in the null space of L_{G_i}. Thus, this π does not show that $G \cong_s^\alpha H$. ☐

Lemma 4.7. *Suppose $G = G_1 + G_2 + ... + G_k$ and $H = H_1 + H_2 + ... + H_k$ are graphs with the same components, and $G \cong_s^\alpha H$. If $\forall i \neq j, |V_{G_i}| = |V_{G_j}| \implies G_i \cong G_j$ or $H_i \cong H_j$, then $\forall i, G_i \cong_s^\alpha H_i$.*

Proof. Let π be some permutation that shows that $G \cong_s^\alpha H$. Consider some arbitrary i. If $V_{\pi(H_i)} = V_{H_i}$, then we know π merely permutes the component H_i amongst itself, so Lemma 4.6 applied with $f = \pi$ gives that $G_i \cong_s^\alpha H_i$. Alternatively, suppose $V_{\pi(H_i)} \neq V_{H_i}$. Since π shows $G \cong_s^\alpha H$ it must ensure G and $\pi(H)$ have the same components. Hence, since G and H have the same components by assumption, we know that H and $\pi(H)$ have the same components. In particular, it must be that $V_{\pi(H_j)} = V_{H_i}$ for some j such that H_i has the same number of vertices as H_j. Now, by assumption, either $H_i \cong H_j$ or $G_i \cong G_j$.

- if $H_j \cong H_i$ by bijection f, then we have that $H_j = f(H_i)$. Hence, $\pi(H_j) = \pi(f(H_i)) = \pi \circ f(H_i)$. Thus, by Lemma 4.7, $G_i \cong_s^\alpha H_i$.
- if $G_i \cong G_j$ by bijection f, then we have that $G_i = f(G_j)$. Now, by a similar argument to Lemma 4.7 using subvectors, it can be shown that $G_i \cong_s^\alpha \pi(H_j)$ via the identity permutation. Hence, $G_i = f(G_j) \cong_s^\alpha f \circ \pi(H_i)$ via the identity permutation. Consequently, $G_i \cong_s^\alpha H_i$ via the permutation $f \circ \pi$.

☐

Lemma 4.8. *If $G = G_1 + G_2 + ... + G_k$ and $H = H_1 + H_2 + ... + H_k$ are graphs with k components, and for each i, $G_i \cong_s^{\alpha_i} H_i$, then $G \cong_s^{\max_i \alpha_i} H$. Furthermore, if G and H have the same components, α_i is optimal for G_i and H_i for each i, and $\forall i \neq j, |V_{G_i}| = |V_{G_j}| \implies G_i \cong G_j$ or $H_i \cong H_j$, then $max_i \alpha_i$ is optimal for G and H.*

Proof. If for each i, $G_i \cong_s^{\alpha_i} H_i$, then $\exists\pi_i\forall x \notin N(L_{G_i}), \frac{1}{\alpha_i} \leq \frac{x^T L_{G_i} x}{x^T L_{\pi_i(H_i)} x} \leq \alpha_i$. Also, we have $N(L_G) = \cap_i N(L_{G_i})$, where we think of each vector in $N(L_{G_i})$ as a vector in R^V where the subvector that is defined over the vertices of G_i must be an element of $N(L_{G_i})$, since a vector is in the Null space of G if and only if it makes each component zero. In addition, define $\pi = \pi_1\pi_2...\pi_k$ be the composition of the permutations for each component. Note, we have G and $\pi(H)$ have the same components since for each i, G_i and $\pi(H_i)$ must have the same components. Now, we have $\forall x \notin N(L_G)$,

$$\frac{x^T L_G x}{x^T L_{\pi(H)} x} = \frac{x^T L_{\sum_{i=1}^k G_i} x}{x^T L_{\sum_{i=1}^k \pi(H_i)} x} = \frac{\sum_{i=1}^k x^T L_{G_i} x}{\sum_{i=1}^k x^T L_{\pi(H_i)} x} \leq \frac{\sum_{i=1}^k \alpha_i x^T L_{\pi_i(H_i)} x}{\sum_{i=1}^k x^T L_{\pi(H_i)} x} \leq \max_i \alpha_i \frac{\sum_{i=1}^k x^T L_{\pi(H_i)} x}{\sum_{i=1}^k x^T L_{\pi(H_i)} x} = \max_i \alpha_i$$

Also, we have,

$$\frac{x^T L_G x}{x^T L_{\pi(H)} x} = \frac{x^T L_{\sum_{i=1}^k G_i} x}{x^T L_{\sum_{i=1}^k \pi(H_i)} x} = \frac{\sum_{i=1}^k x^T L_{G_i} x}{\sum_{i=1}^k x^T L_{\pi(H_i)} x} \geq \frac{\sum_{i=1}^k \frac{1}{\alpha_i} x^T L_{\pi_i(H_i)} x}{\sum_{i=1}^k x^T L_{\pi(H_i)} x} \geq \min_i \frac{1}{\alpha_i} \frac{\sum_{i=1}^k x^T L_{\pi(H_i)} x}{\sum_{i=1}^k x^T L_{\pi(H_i)} x} = \frac{1}{\max_i \alpha_i}$$

Thus, $G \cong_s^{\max_i \alpha_i} H$.

Now, to show the furthermore, note that if $G \cong_s^{\beta} H$ for some smaller β, then $G_i \cong_s^{\beta} H_i$ for any i under the assumptions by Lemma 4.8. In particular, for the i giving the maximum α_i. However, by assumption, α_i is optimal for G_i and H_i, a contradiction. Hence, the function derived is optimal for G and H under these conditions. □

Note that we can apply any result from this section to graphs with the same structure by first applying the permutation that always exists that forces the two graphs to have the same components and applying the result to these graphs. The composition of the two permutations then translates the result to the original graphs.

4.2 Results for Specific Graphs

Note the edge-less graph is only spectrally isomorphic to permutations of itself (which are all the same) and vacuously so.

Proposition 4.9. *$K_n \cong_s^{\frac{n}{n-|V(S)|}} K_n - S$ for any set of edges, S. Furthermore, if S is maximal, then it's optimal.*

Proof. Let $S \subseteq E_{K_n}$. We have that for any permutation, π, that $\pi(K_n - S) = \pi(K_n) - \pi(S) = K_n - \pi(S)$ since every possible edge is present in K_n, so permuting it will not change the edges. Hence, applying Theorem 4.1 to K_n and $K_n - \pi(S)$ with $J = C(K_n - \pi(S)) = K_n - \pi(S) - \{(u, v) \notin \pi(S) | u, v \in V(\pi(S))\}$ we have $\kappa(K_n, K_n - \pi(S)) \leq \frac{n}{n-|V(S)|}$ noting that $|V(S)| = |V(\pi(S))|$. Also, if S is maximal we have that $\pi(S)$ is maximal. Consequently, $C(K_n - \pi(S)) = K_n - \pi(S)$ so each edge has the same max achieved. Hence, the moreover of Theorem 4.1 gives equality for the relative condition number. Since this equality holds over any permutation we have that $\min_\pi \kappa(K_n, \pi(H)) = \frac{n}{n-|V(S)|}$, but this means by definition that $\frac{n}{n-|V(S)|}$ is optimal. □

Proposition 4.10. *For n a power of 3, $\frac{n}{3}C_3 \cong_s^3 \frac{n}{3}P_3$ is optimal*

Proof. Notice that $C_3 = K_3$ and $P_3 = K_3 - (u, v)$ where u and v are the endpoints of the path. Hence, by Proposition 4.9, $C_3 \cong_s^3 P_3$ and 3 is optimal. Hence, since each component of $\frac{n}{3}C_3$ is the same and $\frac{n}{3}C_3, \frac{n}{3}P_3$ have the same components, we can apply Lemma 4.8 to see that $\frac{n}{3}C_3 \cong_s^3 \frac{n}{3}P_3$ is optimal. □

Next, we wish to explore what happens when we delete many edges incident to one vertex. Consequently, We introduce the notation $S_u(f)$ to be an arbitrary set of $f(n)$ edges incident to the vertex u in G. In other words, $S_u(f) = \{(u, v) \in E\}$ and $|S_u(f)| = f(n)$.

Proposition 4.11. $K_n \cong_s^3 K_n - S_u(\lfloor \frac{n}{2} \rfloor - 1)$

Proof. Note that since we deleted $\lfloor \frac{n}{2} \rfloor - 1$ edges from K_n and the degree of u is n-1, there remains $\lceil \frac{n}{2} \rceil$ edges incident to u in the resulting graph. Consider the set of disjoint length 2 paths in K_n that start with u and end with an endpoint of an edge in $S_u(\lfloor \frac{n}{2} \rfloor)$ that exists since the degree of u is n-1 and so there are $\lceil \frac{n}{2} \rceil$ edges remaining after deletion of the others. □

Proposition 4.12. $K_n \cong_s^2 K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Proof. We know that $\lambda_n(K_n) = n$, $\lambda_2(K_{\frac{n}{2}, \frac{n}{2}}) = \frac{n}{2}$, and $K_{\frac{n}{2}, \frac{n}{2}} \subseteq K_n$. Hence, by Lemma 4.8, we know that $K_n \cong_s^2 K_{\frac{n}{2}, \frac{n}{2}}$. □

Proposition 4.13. $K_n \cong_s^n S_n$.

Proof. We know that $\lambda_n(K_n) = n$, $\lambda_2(S_n) = 1$, and $S_n \subseteq K_n$. Hence, by lemma 4.8 we know $K_n \cong_s^n S_n$. $\square$

Conjecture: $C_n \cong_s^4 P_n$

Proof. (Idea) First, label the vertices of P_n from right to left 1 to n, and similarly label the vertices of C_n by taking P_n and adding edge $(1, n)$. Now, consider the permutation:

$$\pi(i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd} \\ (n+1) - \frac{i}{2} & \text{if } i \text{ is even} \end{cases}$$

This permutation ensures that $\forall (u,v) \in E_{C_n}, d_{\pi(P_n)}(u,v) \leq 2$, since for each edge of C_n of form $(i, i+1)$ where $i < \frac{n}{2}$ we have in $\pi(P_n)$ the path $i- > (n+1) - i- > i+1$ and for each edge $(i, i-1)$ for $i \geq \frac{n}{2}$ we have in $\pi(P_n)$ the path $i- > n - i + 2- > i - 1$. Hence, by Lemma 4.4 of [1], the claim holds. $\square$

5 Lower Bounds

5.1 Results for General Graphs

We define the min cut of a disconnected graph as the minimum of each component's min cut. We will present necessary conditions for two graphs to be α-spectrally isomorphic.

Theorem 5.1. *If $G \cong_s^\alpha H$, then $\alpha \geq \max\{\frac{MaxCut(G)}{MaxCut(H)}, \frac{MinCut(G)}{MinCut(H)}, \frac{MaxCut(H)}{MaxCut(G)}, \frac{MinCut(H)}{MinCut(G)}\}$*

Proof. Suppose $S \subset V_G$ induces the max cut of G. Consider the characteristic vector x_S that is 1 for each vertex of S and 0 otherwise. Then, by definition of $x^T L_G x$, we have that $x_S^T L_G x_S = |E_G[S, \overline{S}]|$. Similarly, $x_S^T L_{\pi(H)} x_S = |E_{\pi(H)}[S, \overline{S}]| = |E_H[\pi(S), \overline{\pi(S)}]|$. Hence, since α must be larger than $\frac{x^T L_G x}{x^T L_{\pi(H)} x}$ for all x not in the null space of L_G, it must also be at least $\frac{x_S^T L_G x_S}{x_S^T L_{\pi(H)} x_S} = \frac{|E_G[S,\overline{S}]|}{|E_H[\pi(S),\overline{\pi(S)}]|} \geq \frac{MaxCut(G)}{MaxCut(H)}$. Similarly, let $T \subset V_H$ induce the minimum cut in H and consider the characteristic vector $x_{\pi^{-1}(T)}$. Then, $\frac{x_{\pi^{-1}(T)}^T L_G x_{\pi^{-1}(T)}}{x_{\pi^{-1}(T)}^T L_{\pi(H)} x_{\pi^{-1}(T)}} = \frac{|E_G[\pi^{-1}(T), \overline{\pi^{-1}(T)}]|}{|E_H[\pi(\pi^{-1}(T)), \overline{\pi(\pi^{-1}(T))}]|} = \frac{|E_G[\pi^{-1}(T), \overline{\pi^{-1}(T)}]|}{|E_H[T, \overline{T}]|} \geq \frac{MinCut(G)}{MinCut(H)}$. Now, since $G \cong_s^\alpha H \iff H \cong_s^\alpha G$, we can apply these two results to H and G to get α is at least each of these quantities and so its at least the max of them all. $\square$

Theorem 5.2. *If $G \cong_s^\alpha H$, then $\alpha \geq \max\{\frac{\Delta(G)}{\Delta(H)}, \frac{\delta(G)}{\delta(H)}, \frac{\Delta(H)}{\Delta(G)}, \frac{\delta(H)}{\delta(G)}\}$.*

Proof. Consider the vector δ_u that is one at u and 0 otherwise, where u is some vertex of G with maximum degree. Then, we have that $\delta_u^T L_G \delta_u = \sum_{(v,w) \in E(G)} (\delta_u(v) - \delta_u(w))^2 = \sum_{(u,v) \in E(G)} (\delta_u(u) - \delta_u(v))^2 = \sum_{(u,v) \in E(G)} 1 = d_G(u)$. Similarly, $\delta_u^T L_{\pi(H)} \delta_u = d_H(\pi(u))$. Hence, since α must be at least the ratio of the Rayleigh quotients for any vector that is not in the null space of the two Laplacians and since we have found a particular vector that gives the ratio of Rayleigh quotients a value of $\frac{d_G(u)}{d_H(\pi(u))}$, it must be the case that $\alpha \geq \frac{d_G(u)}{d_H(\pi(u))} = \frac{\Delta(G)}{d(\pi_H(u))} \geq \frac{\Delta(G)}{\Delta(H)}$. Similarly, if we consider δ_u where $\pi(u)$ has minimum degree in H, then we have $\alpha \geq \frac{d_G(u)}{d_H(\pi(u))} = \frac{d_G(u)}{\delta(H)} \geq \frac{\delta(G)}{\delta(H)}$. Hence, α is at least the largest of the two quantities. Now, we know by symmetry of $\cong_s$ that $G \cong_s^\alpha H \iff H \cong_s^\alpha G$. Hence, we can repeat the proof with H,G to get the claim. $\square$

Now we show that under relatively mild conditions, the above claim strengthens even further.

Lemma 5.3. *If $G \cong_s^\alpha H$ and there is some permutation, π, of H demonstrating this fact satisfying that there exists a vertex u of maximum degree in G and $\pi(u)$ has minimum degree in H, then $\alpha \geq \frac{\Delta(G)}{\delta(H)}$. Alternatively, if there is some S that induces a maximum cut of G with $\pi(S)$ being a minimum cut in H, then $\alpha \geq \frac{MaxCut(G)}{MinCut(H)}$.*

Proof. Since we know such a u exists, plugging δ_u into the ratio of Rayleigh quotients gives a value of $\frac{\Delta(G)}{\delta(H)}$ by the proof of the Theorem 5.2, so α must be at least this quantity. Similarly, if we consider the characteristic vector for S, we see the second claim holds. □

Corollary 5.3.1. *If G or H is regular, then $\alpha \geq \frac{\Delta(G)}{\delta(H)}$. Also, if G is d-regular and H is k-regular, then $\alpha \geq \frac{d}{k}$.*

Proof. If H is d-regular, every vertex has degree d, which is the minimum degree of H. In particular, given any vertex u of maximum degree in G, we have that for any permutation, π, applied to H that $\pi(u)$ has minimum degree. Thus, by Lemma 5.3 we are done. On the other hand, if G is d-regular, then every vertex of G has maximum degree. Thus, if u is any vertex of minimum degree in H, $\pi^{-1}(u)$ satisfies the conditions of the Lemma and again we are done. Now, if both are regular as described above, then we know by the Theorem that $\alpha \geq \frac{d}{k}$. □

Lemma 5.4. *If $G \cong_s^\alpha H$ and G has k components, then $\alpha \geq \max\{\frac{\lambda_{k+1}(G)}{\lambda_n(H)}, \frac{\lambda_{k+1}(H)}{\lambda_n(G)}\}$*

Proof.

$$\alpha \geq \max_x \frac{x^T L_G x}{x^T L_{\pi(H)} x} \geq \min_x \frac{x^T L_G x}{x^T L_{\pi(H)} x} = \min_x \frac{x^T L_G x}{x^T L_{\pi(H)} x} \times \frac{\frac{1}{x^T x}}{\frac{1}{x^T x}} = \min_x \frac{\frac{x^T L_G x}{x^T x}}{\frac{x^T L_{\pi(H)}}{x^T x}} \geq \frac{\min_x \frac{x^T L_G x}{x^T x}}{\max_x \frac{x^T L_{\pi(H)} x}{x^T x}} = \frac{\lambda_{k+1}(G)}{\lambda_n(H)}$$

Now, repeating the argument and applying symmetry of $\cong_s^\alpha$ completes the proof. □

5.2 Results for Specific Graphs

Now, using the previous facts we can show that certain graphs are not constant-spectrally isomorphic.

Proposition 5.5. *The following lower bounds hold for the common graphs:*

1. $K_n \cong_s^\alpha S_n \implies \alpha \geq n-1$
2. $K_n \cong_s^\alpha C_n \implies \alpha \geq \frac{n-1}{2}$
3. $S_n \cong_s^\alpha C_n \implies \alpha \geq \frac{n-1}{2}$
4. $S_n \cong_s^\alpha P_n \implies \alpha \geq \frac{n-1}{2}$

Proof.

1. By Corollary 5.3.1 we have since K_n is regular that $\alpha \geq \frac{n-1}{1} = n-1$.
2. Again, by Corollary 5.3.1 we have that $\alpha \geq \frac{n-1}{2}$.
3. Here, we just apply Theorem 5.2 to see that $\alpha \geq \frac{n-1}{2}$.

4. Lastly, by Theorem 5.2 we see that $\alpha \geq \frac{n-1}{2}$.

□

Notice the last two graphs actually exhibit an interesting general result.

Proposition 5.6. *Two trees need not be constant-spectrally isomorphic.*

Proof. The star and path are both trees, yet by the previous Proposition we know they are not constant-spectrally isomorphic. □

Proposition 5.7. *if n is even, $K_n \cong_s^\alpha P_n \implies \alpha \geq \frac{n^2}{2}$*

Proof. Consider any permutation, π, of P_n. Consider the cut in $\pi(P_n)$ that contains the left half of the path. Then, just as in the proof of Theorem 5.1, we have α is lower bounded by the size of the cut in K_n, which is $\frac{n^2}{2}$ since for each of the $\frac{n}{2}$ vertices in the cut, there are exactly that many edges to that vertex that cross the cut, divided by the size of the cut in $\pi(P_n)$, which is 1. □

Proposition 5.8. *$K_n \cong_s^2 K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is optimal if n is odd.*

Proof. Proposition 4.12 gives that the relation holds. Also, when n is odd, we have that the minimum degree of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is exactly $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Hence, by Theorem 5.2 we know that any β for which these two graphs are spectrally isomorphic must be at least $\frac{n-1}{\frac{n-1}{2}} = 2$. Hence, 2 is optimal. □

Proposition 5.9. *$K_n \cong_s^n S_n$ and n is nearly optimal, and may be optimal.*

Proof. The relation holds by Proposition 4.13. Also, Proposition 5.5 gives that the smallest α that could possibly work is $n-1$. Hence, n is nearly optimal, and may in fact be the best possible. □

We can also determine conditions for which a permutation may show that $C_n \cong_s^\alpha P_n$.

Proposition 5.10. *Suppose $C_n \cong_s^\alpha P_n$ and π is some permutation that shows this to be true. Then, $\forall (u,v) \in E_{C_n}$, $d_H(\pi(u), \pi(v)) \leq \alpha$.*

Proof. We show the contrapositive. Suppose $\exists (u,v) \in E_G$ with $d_{\pi(H)}(u,v) > \alpha$. Let $u = p_0, p_1, ..., p_{d_{\pi(H)}(u,v)} = v$ be this path in $\pi(H)$. Then, consider the vector x that satisfies $x(p_i) = i$ for each $i \in [d_{\pi(H)}(u,v)]$ and $x(w) = 0$ for all other vertices. Then, we have that

$$\frac{x^T L_G x}{x^T L_{\pi(H)} x} = \frac{(0 - d_{\pi(H)}(u,v))^2}{\sum_{i=1}^{d_{\pi(H)}(u,v)} 1^2} = \frac{d_{\pi(H)}(u,v)^2}{d_{\pi(H)}(u,v)} = d_{\pi(H)}(u,v) > \alpha$$

□

6 Complexity Theoretic Results

6.1 Spectral Complexity

Proposition 6.1. $CSI \subseteq LSI \subset PSI$

Proof. The containments immediately follow by definition. The last containment is strict since by Proposition 5.7 there exists a pair of graphs so that the graphs are in PSI but not in LSI. □

Proposition 6.2. $\forall \alpha < n-1, n \leq \beta < \frac{n^2}{2}, SI(\alpha) \subseteq SI(n) \subset SI(\beta) \subset PSI$

Proof. Proposition 5.9 gives that $(K_n, S_n) \in SI(n) \setminus SI(\alpha)$ for any $\alpha < n-1$. Also, Proposition 5.7 gives a pair of graphs that are in PSI but not in $SI(\beta)$ for any $\beta < \frac{n^2}{2}$. □

Proposition 6.3. $CED \neq CSI$

Proof. We know by Proposition 4.12 that there exists two graphs that are constant-spectrally isomorphic, yet differ by more than a constant number of edges. □

6.2 General Complexity

Let $GD(\alpha) = \{(G, H) | G\alpha - dominates H\}$.

Proposition 6.4. $SI(\alpha) \subseteq GD(\alpha)$

Proof. $G \cong_s^\alpha H \implies$ G α-dominates H. □

Proposition 6.5. $Graph - Isomorphism = SI(1)$

Proof. We know that G is isomorphic to H if and only if $G \cong_s^1 H$. Consequently, the equality holds. □

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